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On a Generalization of Identities of Hoggatt and Horadam

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It can be established for the sequence of Fibonacci numbers $\{F_n\}$, that

$$F_n^2 + F_{n-1}^2 = F_{2n-1}. \tag{1}$$

This is actually a particular case of slightly rearranged forms of identities (I19) of Hoggatt (1969) and (4.2) of Horadam (1965),

$$F_n^2 - (-1)^{n+k}F_k^2 = F_{n-k}F_{n+k}, \tag{2}$$

$$w_n^2 - qw_{n-1}^2 = aw_{2n} + (b-pa)w_{2n-1}, \tag{3}$$

in which $\{w_n\} \equiv \{w_n(a,b;p,q)\}$ is a generalized second order sequence with initial conditions $w_0 = a$, $w_1 = b$ and defined by the linear homogenous recurrence relation

$$w_n = pw_{n-1} - qw_{n-2}, (n \geq 2). \tag{4}$$

It can be seen that the ordinary Fibonacci numbers are the particular case given by

$$F_{n+1} = w_n(1,1;1,-1). \tag{5}$$

It can also be seen that (1) comes from (2) when $k = n-1$, and from (3) with the appropriate changes from (4) and (5). A stronger generalization is given by

$$w_n^2 - qw_{n-1}^2 = w_r w_{2n-r} - qw_{r-1}w_{2n-r-1}. \tag{6}$$

The result in (6) can be proved by induction on r. When $r = 0$, we get the result in (3). Assume the result is true for $r = 1,2,..,k-1$. Then from the inductive hypothesis and repeated use of (4):

$$\begin{aligned} w_n^2 - qw_{n-1}^2 &= w_{k-1}w_{2n-k+1} - qw_{k-2}w_{2n-k} \\ &= w_{k-1}(pw_{2n-k} - qw_{2n-k-1}) - qw_{k-2}w_{2n-k} \\ &= (pw_{k-1} - qw_{k-2})w_{2n-k} - qw_{k-1}w_{2n-k-1} \\ &= w_k w_{2n-k} - qw_{k-1}w_{2n-k-1}, \text{ as required.} \end{aligned}$$

References

Hoggatt, Verner E Jr. 1969. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin.
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