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NEW VARIANT OF A FIBONACCI PLANE

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The concept of the Fibonacci square was introduced in [1] and generalized in [2] to some forms of Fibonacci planes and in [3] to a Fibonacci spaces. Here we shall describe one more possible generalization of the Fibonacci plane. It has the form in Fig. 1. Immediately we can see that it is a direct generalization of the positive Fibonacci plane from [2].

Let us denote:

$$\begin{aligned} F(1,-1) &= a \\ F(1,1) &= b, \\ F(-1,-1) &= c, \\ F(-1,1) &= d, \end{aligned} \tag{1}$$

and for every two integers m, n let $F(m,n)$ denote the number which stays in the place with "coordinates" (with the sense of (1) and Fig. 1) m and n. Then for $m, n \geq 1$ we have:

$$\begin{aligned} F(1,n) &= f_{n-1}.a + f_n.b, \\ F(-1,n) &= f_{n-1}.c + f_n.d, \\ F(1,-n) &= f_n.a + f_{n-1}.b, \\ F(-1,-n) &= f_n.c + f_{n-1}.d, \\ F(m,1) &= f_m.b + f_{m-1}.d, \\ F(m,-1) &= f_m.a + f_{m-1}.c, \\ F(-m,1) &= f_{m-1}.b + f_m.d, \\ F(-m,-1) &= f_{m-1}.a + f_m.c \end{aligned}$$

THEOREM 1: For all natural numbers $m, n \geq 1$:

$$\begin{aligned} &(a) F(m,n) = f_m.f_{n-1}.a + f_m.f_n.b + f_{m-1}.f_{n-1}.c + f_{m-1}.f_n.d, \\ &(b) F(m,-n) = f_m.f_n.a + f_m.f_{n-1}.b + f_{m-1}.f_n.c + f_{m-1}.f_{n-1}.d, \\ &(c) F(-m,n) = f_{m-1}.f_{n-1}.a + f_{m-1}.f_n.b + f_m.f_{n-1}.c + f_m.f_n.d, \\ &(d) F(-m,-n) = f_{m-1}.f_n.a + f_{m-1}.f_{n-1}.b + f_m.f_n.c + f_m.f_{n-1}.d, \end{aligned}$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\cdots$	$9a+6b+6c+4d$	$6a+3b+4c+2d$	$3a+3b+2c+2d$	$3a+2c$	$3b+2d$	$3a+3b+2c+2d$	$3a+6b+2c+4d$	$6a+9b+4c+6d$	$\cdots$
$\cdots$	$6a+4b+3c+2d$	$4a+2b+2c+d$	$2a+2b+c+d$	$2a+c$	$2b+d$	$2a+2b+c+d$	$2a+4b+c+2d$	$4a+6b+2c+3d$	$\cdots$
$\cdots$	$3a+2b+3c+2d$	$2a+b+2c+d$	$a+b+c+d$	$a+c$	$b+d$	$a+b+c+d$	$a+2b+c+2d$	$2a+3b+2c+3d$	$\cdots$
$\cdots$	$3a+2b$	$2a+b$	$a+b$	$\mathbf{a}$	$\mathbf{b}$	$a+b$	$a+2b$	$2a+3b$	$\cdots$
$\cdots$	$3c+2d$	$2c+d$	$c+d$	$\mathbf{c}$	$\mathbf{d}$	$c+d$	$c+2d$	$2c+3d$	$\cdots$
$\cdots$	$3a+2b+3c+2d$	$2a+b+2c+d$	$a+b+c+d$	$a+c$	$b+d$	$a+b+c+d$	$a+2b+c+2d$	$2a+3b+2c+3d$	$\cdots$
$\cdots$	$3a+2b+6c+4d$	$2a+b+4c+2d$	$a+b+2c+2d$	$a+2c$	$b+2d$	$a+b+2c+2d$	$a+2b+2c+4d$	$2a+3b+4c+6d$	$\cdots$
$\cdots$	$6a+4b+9c+6d$	$4a+2b+6c+3d$	$2a+2b+3c+3d$	$2a+3c$	$2b+3d$	$2a+2b+3c+3d$	$2a+4b+3c+6d$	$4a+6b+6c+9d$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let us also define for the above a, b, c, d and for $m, n \geq 1$:

$$F(m,n) = F_a(m,n).a + F_b(m,n).b + F_c(m,n).c + F_d(m,n).d, \tag{2}$$

where $F_a(m,n), F_b(m,n), F_c(m,n), F_d(m,n)$ are the coefficients before a, b, c, d in (2), respectively. It can be seen easily that the following equalities are valid for all natural numbers m, n:

$$\begin{aligned} F_a(m,-n) &= F_b(m,n), \\ F_b(m,-n) &= F_a(m,n), \\ F_c(m,-n) &= F_d(m,n), \\ F_d(m,-n) &= F_c(m,n), \\ F_a(-m,n) &= F_c(m,n), \\ F_b(-m,n) &= F_d(m,n), \\ F_c(-m,n) &= F_a(m,n), \\ F_d(-m,n) &= F_b(m,n). \end{aligned} \tag{3}$$

Now, from the equalities (3) the next equalities also hold

$$\begin{aligned} F_a(-m,-n) &= F_d(m,n), \\ F_b(-m,-n) &= F_c(m,n), \\ F_c(-m,-n) &= F_b(m,n), \\ F_d(-m,-n) &= F_a(m,n). \end{aligned}$$

THEOREM 2: For every natural number $n \geq 1$:

$$\sum_{(i,j)\in\{-n,n\}} F(i,j) = f_{n+1}^2.(a+b+c+d).$$

Proof: From (2) it follows that

$$\begin{aligned} &\sum_{(i,j)\in\{-n,n\}} F(i,j) \\ &= (2.f_n.f_{n-1} + f_n.f_n + f_{n-1}.f_{n-1}).(a+b+c+d) \\ &= (f_{n-1} + f_n)^2.(a+b+c+d) \\ &= f_{n+1}^2.(a+b+c+d). \end{aligned}$$

A lot of interesting identities can be proved for the members of the new Fibonacci plane. For example, for every natural number $n \geq 1$ it is valid:

$$F(n,n) - F(n+1,n-1) = (-1)^n(a-b+c),$$

$$F(n,-n) - F(n+1,-n+1) = (-1)^n(-a+b-c),$$

etc. Really, from Theorem 1 (a) it follows that

$$F(n,n) - F(n+1,n_1)$$

$$= f_n.f_{n-1}.a + f_n.f_n.b + f_{n-1}.f_{n-1}.c + f_{n-1}.f_n.d$$

$$-f_{n+1}.f_{n-2}.a - f_{n+1}.f_{n-1}.b - f_n.f_{n-2}.c - f_n.f_{n-1}.d$$

Having in mind the well known identities, for every natural number $p \geq 1$:

$$f_{p+1}.f_{p-1} - f_p^2 = (-1)^p$$

we obtain

$$F(n,n) - F(n+1,n_1)$$

$$= (f_n.f_{n-1} - f_{n+1}.f_{n-2}).a + (f_n.f_n - f_{n+1}.f_{n-1}).b(f_{n-1}.f_{n-1} - f_n.f_{n-2}).c + (f_{n-1}.f_n - f_n.f_{n-1}).d$$

$$= (-1)^n.a + (-1)^{n-1}.b + (-1)^n.c$$

$$= (-1)^n.(a - b + c).$$

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