NNTDM 8 (2002) 3, 107-111

AN INVARIANT INTEGRALS IN THE p-ADIC NUMBER FIELDS

LEE CHAE JANG

Department of Mathematics and Computater Science
KonKuk University, Chungju 380-701, S. Korea
e-mail:leechae.jang@kku.ac.kr

ABSTRACT. In this paper we investigate some properties of non-Archimedean integration which is defined by T. Kim, cf. [2]. By using our results in this paper, we can give an answer of the problems which is remained by I.-C. Huang and S-Y. Huang in [1: p. 179]

§1. INTRODUCTION

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$.

Let l be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/lp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<lp \\ (a,p)=1}} (a + lp\mathbb{Z}_p),$$

$$a + lp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{lp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < lp^N$.

For any positive integer N, we set

$$\mu_1(a + lp^N\mathbb{Z}_p) = \frac{1}{lp^N}$$

and this can be extended to a distribution on X (see [2]).

1991 *Mathematics Subject Classification.* 11B68, 11S80.
Key words and phrases. p-adic integrals, Euler number, Non-Archimeden Integration.

This distribution yields an integral for non-negative integer m :

$$\int_X x^m \, d\mu_1(x) = B_m,$$

where B_m are called usual Bernoulli numbers.

The Euler numbers E_m are defined by the generating function in the complex number field as follows:

$$\frac{2}{e^t+1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad (|t| < \pi), \tag{1}$$

where we use the technique method notation by replacing E^m by E_m $(m \geq 0)$, symbollically, cf. [2], [4], [5], [6], [7].

The Bernoulli numbers with order k, $B_n^{(k)}$, were defined by

$$\left(\frac{t}{e^t-1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \text{ cf. [5], [6].} \tag{2}$$

Let u be algebraic in complex number field . Then Frobenius-Euler numbers were defined by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \text{ cf. [2], [3].} \tag{3}$$

By (1), (3), note that $H_n(-1) = E_n$.

In this paper, we will give the interesting formulae for sums of products of Euler numbers (= Frobenius-Euler numbers) by using p-adic Euler integration which is defined by K. Shiratani-S. Yamamoto, (see [3]). Our result is an answer of the problem which is remained by I. C. Huang and S.-Y. Huang in [1: p. 179].

§2. Sums of products of Euler numbers

Let $u \in \mathbb{C}_p$ with $|1-u^f|_p \geq 1$ for each positive integer f. Then the p-adic Euler measure was defined by

$$E_u(x) = E_u(x + dp^N \mathbb{Z}_p) = u^{dp^N - x}/(1 - u^{dp^N}), \text{ cf. [3].}$$

Now, we define Euler polynomials with order n by

$$\left(\frac{u}{1-u}\right)^m H_n^{(m)}(u,x) = \underbrace{\int_X \cdots \int_X}_{m \text{ times}} (x + x_1 + \cdots + x_m)^n dE_u(x_1) \cdots dE_u(x_m). \tag{4}$$

In the case $x=0$, we use the following notations :

$H_n^{(k)}(u,0)=H_n^{(k)}(u)$, $H_n^{(1)}(u)=H_n(u)$, cf. [4: p. 77].

In [3], K. Shiratani-S.Yamamoto have given the following formula:

$$\int_{\mathbb{Z}_p} x^n dE_u(x)=\frac{u}{1-u}H_n(u). \tag{5}$$

By (4), (5), we easily see that $\lim_{k\to 1} H_n^{(k)}(u)=H_n(u)$.

For any positive integers m, $H_n^{(m)}(u,x)$ can be written by

$$H_n^{(m)}(u,x)=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}H_j^{(m)}(u).$$

We may now mention the following formulae which are easy to prove:

$$\left(\frac{u}{1-u}\right)^m H_n^{(m)}(u,x)=l^n\sum_{l_1,\cdots,l_m=0}^{l-1}\frac{u^{ml-\sum_{i=1}^m l_i}}{(1-u^l)^m}H_n^{(m)}(u^l,\frac{x+l_1+\cdots+l_m}{l}), \tag{6}$$

where

$$\sum_{l_1,\cdots,l_m=0}^{l-1}=\sum_{l_1=0}^{l-1}\sum_{l_2=0}^{l-1}\cdots\sum_{l_m=0}^{l-1}.$$

For $m=1$, Eq.(6) is same the result of K. Shiratani-S.Yamamoto (see [3]).

By using (4) and multinomial coefficients, We obtain the following theorem:

Theorem. *For $\alpha_1,\alpha_2,\cdots,\alpha_m\in\mathbb{C}_p$ and positive integer n,m, we have*

$$H_n^{(m)}(u,\alpha_1+\alpha_2+\cdots+\alpha_m)=\sum_{\substack{i_1,\cdots,i_m\\ n=i_1+\cdots+i_m}}\binom{n}{i_1,\cdots,i_m}H_{i_1}(u,\alpha_1)H_{i_2}(u,\alpha_2)\cdots H_{i_m}(u,\alpha_m),$$

where $\binom{n}{i_1,\cdots,i_m}$ is the multinomial coefficient.

Remark. The above theorem is an answer of the problem which was remained in [1: p. 179].

Remark. Note that $H_n(-1)=\sum_{k=0}^{n}\binom{n+1}{k}2^k B_k$, where B_k are the kth ordinary Bernoulli numbers.

Remark. By using Volkenborn integral, it was well known that

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}x^n d\mu_1(x)\frac{t^n}{n!},\ \text{cf. [2, 3]}.$$

In [4], note that

$$\left(\frac{t}{e^t-1}\right)^k=\sum_{n=0}^{\infty}\underbrace{\int_X\int_X\cdots\int_X}_{k \text{ times}}(x+x_1+\cdots+x_k)^n d\mu_1(x_1)d\mu_1(x_2)\cdots d\mu_1(x_k)\frac{t^n}{n!}. \quad (7)$$

The Bernoulli polynomials with order k, $B_n^{(k)}(x)$, were defined by

$$B_n^{(k)}(x)=\underbrace{\int_X\int_X\cdots\int_X}_{k \text{ times}}(x+x_1+\cdots+x_k)^n d\mu_1(x_1)d\mu_1(x_2)\cdots d\mu_1(x_k), \text{ cf. [3], [4], [6].}$$

In the case $x=0$, we write $B_n^{(k)}(0)=B_n^{(k)}$, cf. [4].

In [1], the authors proved the formulae of sums of products of Bernoulli numbers of higher order by using theory of residues. By using the properties of invariant p-adic integrals in this paper, we can also give the same formulae on the sums of products for $B_n^{(k)}$ in [1].

Let χ be a Dirichlet character with conductor f. We set $p^*=p$ for $p\geq 2$, and $p^*=4$ for $p=2$.

Let $\bar{f}=(f,p^*)$ be denoted by the least common multiple of the conductor f of χ and p^*.

Now, we define the generalized Bernoulli numbers of higher order with χ as

$$B_{n,\chi}^{(m)}=\int_X\cdots\int_X\chi(x_1+\cdots+x_m)(x_1+\cdots+x_m)^n d\mu_1(x_1)\cdots d\mu_1(x_m). \quad (8)$$

We easily get in (8)

$$B_{n,\chi}^{(m)}=l^{n-m}\sum_{x_1,\cdots,x_m=0}^{l-1}B_n^{(m)}\left(\frac{x_1+\cdots+x_m}{l}\right)\chi(x_1+\cdots+x_m).$$

where $B_{n,\chi}$ is the generalized ordinary Bernoulli number with χ.

By (8), we have

$$B_{n,\chi}^{(m)}=\lim_{\rho\to\infty}\frac{1}{(\bar{f}p^\rho)^m}\sum_{1\leq x_1\leq \bar{f}p^\rho}\cdots\sum_{1\leq x_m\leq \bar{f}p^\rho}\chi(x_1+\cdots+x_m)(x_1+\cdots+x_m)^n.$$

The investigation of these numbers are left to the interested reader.

ACKNOWLEDGEMENTS: The author wishes to express his sincere gratitute to the referees for their valuable comments.

REFERENCES

1. I.-C. Huang, S.-Y. Huang, *Bernoulli numbers and polynomials via residues*, J. Number Theory **76** (1999), 178-193.
2. T. Kim, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory **76** (1999), 320-329.
3. T. Kim, L.C. Jang and H. K. Pak, *A note on q-Euler numbers and Genocchi numbers*, Proc. Japan Acad. Ser A Math. Sci. **77** (2001), 139-141.
4. K. Shiratani and S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ., **39** (1985), 113-125.
5. H.M. Srivastava, *Some formulas for the Bernoulli and Euler polynomials at rational arguments*, Math. Proc. Cambridge Philos. Soc., **129** (2000), 77-84.
6. H.M. Srivastava and P.G. Todorov, *An explicit formula for the generalized Bernoulli polynomials*, J. Math. Anal.Appl., **130** (1988), 509-513.
7. H.M. Srivastava, Series Associated with the Zeta and Related Functions, *Kluwer Acad Publishers (Dordreht/Boston/London)*, 2001.