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Powers as a Difference of Squares: The Effect on Triples

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Abstract

All powers equal a difference of squares so that triples may be expressed as the sum of three squares which equal the sum of another three squares. However, when $n > 2$ integer values for all the components should be impossible according to the work which peaked with Wiles. By utilising the properties of the Modular Ring $\mathbb{Z}_4$ we illustrate how the underlying Class structures of the integers justifies this constraint.

1. Introduction

We have previously shown (Leyendekkers and Shannon, 2001) that for $N \in \mathbb{Z}$

$$N^3 = x^2 - y^2 \tag{1.1}$$

where $(x,y) = (\tfrac{1}{2}N(N+1), \tfrac{1}{2}N(N-1))$; for N even, we also have $(x,y) = (\tfrac{1}{4}N(N+4), \tfrac{1}{4}N(N-4))$, with $N = x - y$.
Since

$$\begin{aligned} N^n &= N^{n-3}N^3 \\ &= N^{n-3}(x_N^2 - y_N^2) \qquad (1.2) \\ &= (N^{\frac{1}{2}(n-3)}x_N)^2 - (N^{\frac{1}{2}(n-3)}y_N)^2. \qquad (1.3) \end{aligned}$$

Consider the triple

$$c^n - b^n = a^n \tag{1.4}$$

with

$$\begin{aligned} c^n &= X_c^2 - Y_c^2, \\ b^n &= X_b^2 - Y_b^2, \\ a^n &= X_a^2 - Y_a^2, \end{aligned}$$

with

$$\begin{aligned} X_z &= z^{\frac{1}{2}(n-3)}x_z, \\ Y_z &= z^{\frac{1}{2}(n-3)}y_z, \end{aligned}$$

in which $z \in \{a,b,c\}$. Thus Equation (1.4) may be expressed as

$$X_c^2 + Y_b^2 + Y_a^2 = X_a^2 + Y_c^2 + X_b^2. \tag{1.5}$$

Let $P(x,y,z)$ and $P'(x',y',z')$ be any two points on the surface of a sphere with centre $C(0,0,0)$ and radius d. Then the equation of the sphere is

$$x^2 + y^2 + z^2 = d^2 = x'^2 + y'^2 + z'^2$$

which is a geometric expression of the fact that the sum of three squares can equal the sum of another three squares. This paper examines the foundations for powers higher than 2 by considering the parity structures in the modular rings. For related research the reader is referred to the work of Dujella (1996), Ewell (1992) and Zia (1991).

Let

$$c_1^2 - b_1^2 = a_1^2 \tag{1.6}$$

and

$$c_2^2 - b_2^2 = a_2^2 \tag{1.7}$$

represent two primitive Pythagorean triples (pPts). Then, on subtracting Equation (1.7) from Equation (1.6) we obtain

$$c_1^2 + b_2^2 + a_2^2 = c_2^2 + b_1^2 + a_1^2, \tag{1.8}$$

which will be true for any two pPts.

Since c and a are odd and b even (that is, the pPt will always have two odd and one even component), the parity pattern for Equation (1.8) is

$$\text{odd} + \text{even} + \text{odd} = \text{odd} + \text{even} + \text{odd}. \tag{1.9}$$

Thus the sum of three integer squares can equal the sum of another three integer square with a parity pattern of *odd, odd, even.*

As will be shown below, Equation (1.5) cannot have these parity groupings, which, in turn, is compatible with the fact that Equation (1.4) cannot have integer solutions when $n > 2$. However, square-triple equalities (STE) can be derived from non-pPt relationships; that is,

$$c_1^2 - b_1^2 \neq a_1^2,$$
$$c_2^2 - b_2^2 \neq a_2^2,$$

but

$$(c_1^2 - b_1^2) - (c_2^2 - b_2^2) = a_1^2 - a_2^2. \tag{1.10}$$

Thus, we need to analyse why Equation (1.5) is incompatible with this type of STE as well.

2. Power Structure within the Modular Ring $\mathbb{Z}_4$

For notational convenience within the modular ring $\mathbb{Z}_4$ we identify the integers by $(4r_i + i)$ where $\bar{i}$ is the Class and r_i the row in a tabular array of $\bar{i}$. Obviously, even integers $\in \{\bar{0}_4, \bar{2}_4\}$ and odd integers $\in \{\bar{1}_4, \bar{3}_4\}$. There are no powers in $\bar{2}_4$ and no even powers in $\bar{3}_4$.

Odd Powers

Other properties have been discussed in detail in Leyendekkers, Rybak and Shannon (1997) where it is shown that the Class structures for (c, b, a) which are permissible for Equation (1.4) are:

$$(\bar{1}_4, \bar{0}_4, \bar{1}_4), (\bar{1}_4, \bar{2}_4, \bar{1}_4), (\bar{0}_4, \bar{1}_4, \bar{3}_4), (\bar{2}_4, \bar{1}_4, \bar{3}_4).$$

Thus odd $c \in \overline{1}_4$, whereas odd $a \in \{\overline{1}_4, \overline{3}_4\}$. Even $c \in \{\overline{0}_4, \overline{2}_4\}$ and odd $b \in \overline{1}_4$, but even $b \in \{\overline{0}_4, \overline{2}_4\}$. Table 1 shows the Class structure of the various components of Equation (1.5). The x,y class structure in Table 2 is from Leyendekkers and Shannon (2001).

Class of z	n *	$z^{½(n-3)}$	x_z	y_z	$X_z = z^{½(n-3)}x_z$	$Y_z = z^{½(n-3)}y_z$
$\overline{1}_4$	$\overline{3}_4$ V	$\overline{1}_4$	$\overline{1}_4$	$\overline{0}_4$	$\overline{1}_4 \times \overline{1}_4 = \overline{1}_4$	$\overline{1}_4 \times \overline{0}_4 = \overline{0}_4$
	$\overline{1}_4$		$\overline{3}_4$	$\overline{2}_4$	$\overline{1}_4 \times \overline{3}_4 = \overline{3}_4$	$\overline{1}_4 \times \overline{2}_4 = \overline{2}_4$
$\overline{3}_4$	$\overline{3}_4$	$\overline{1}_4$	$\overline{2}_4$	$\overline{3}_4$	$\overline{1}_4 \times \overline{2}_4 = \overline{2}_4$	$\overline{1}_4 \times \overline{3}_4 = \overline{3}_4$
			$\overline{0}_4$	$\overline{1}_4$	$\overline{1}_4 \times \overline{0}_4 = \overline{0}_4$	$\overline{1}_4 \times \overline{1}_4 = \overline{1}_4$
	$\overline{1}_4$	$\overline{3}_4$	$\overline{2}_4$	$\overline{3}_4$	$\overline{3}_4 \times \overline{2}_4 = \overline{2}_4$	$\overline{3}_4 \times \overline{3}_4 = \overline{1}_4$
			$\overline{0}_4$	$\overline{1}_4$	$\overline{3}_4 \times \overline{0}_4 = \overline{0}_4$	$\overline{3}_4 \times \overline{1}_4 = \overline{3}_4$
$\overline{0}_4$	$\overline{3}_4$ V	$\overline{0}_4$	$\overline{2}_4$	$\overline{2}_4$	$\overline{0}_4 \times \overline{2}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{2}_4 = \overline{0}_4$
	$\overline{1}_4$		$\overline{0}_4$	$\overline{0}_4$	$\overline{0}_4 \times \overline{0}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{0}_4 = \overline{0}_4$
$\overline{2}_4$	$\overline{3}_4$ V	$\overline{0}_4$	$\overline{1}_4$	$\overline{3}_4$	$\overline{0}_4 \times \overline{1}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{3}_4 = \overline{0}_4$
	$\overline{1}_4$		$\overline{3}_4$	$\overline{3}_4$	$\overline{0}_4 \times \overline{3}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{3}_4 = \overline{0}_4$
	$n > 5$		$\overline{3}_4$	$\overline{1}_4$	$\overline{0}_4 \times \overline{3}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{1}_4 = \overline{0}_4$

Table 1: $z \in \{a, b, c\}$

* If $n = 3, z^{½(n-3)} = 1, X_z = x_z$ and $Y_z = y_z$.

If $n = 5$, and $z \in \overline{2}_4, X_z \in \overline{2}_4, Y_z \in \overline{2}_4$.

N	Class of N	Row of N	Class of (x,y)	Class of (x',y')	$(x,y)/(x',y')$	Rows of (x,y)
1	$\overline{1}_4$	0	$(\overline{1}_4\overline{0}_4)$		(1,0)	(0,0)
2	$\overline{2}_4$	0	$(\overline{3}_4\overline{1}_4)$		(3,1)	(0,0)
3	$\overline{3}_4$	0	$(\overline{2}_4\overline{3}_4)$		(6,3)	1,0
4	$\overline{0}_4$	1	$(\overline{2}_4\overline{2}_4)$	$(\overline{0}_4\overline{0}_4)$	(10,6)	2,1
					(8,0)	2,0
5	$\overline{1}_4$	1	$(\overline{3}_4\overline{2}_4)$		(15,10)	3,2
6	$\overline{2}_4$	1	$(\overline{1}_4\overline{3}_4)$	$(\overline{3}_4\overline{3}_4)$	(21,15)	5,3
		1			(15,3)	3,0
7	$\overline{3}_4$	2	$(\overline{0}_4\overline{1}_4)$		(28,21)	7,5
8	$\overline{0}_4$	2	$(\overline{0}_4\overline{0}_4)$	$(\overline{0}_4\overline{0}_4)$	(36,28)	9,7
					(24,8)	6,2
9	$\overline{1}_4$	2	$(\overline{1}_4\overline{0}_4)$		(45,36)	11,9
10	$\overline{2}_4$	2	$(\overline{3}_4\overline{1}_4)$	$(\overline{3}_4\overline{3}_4)$	(55,45)	13,11
					(35,15)	8,3
11	$\overline{3}_4$	2	$(\overline{2}_4\overline{3}_4)$		(66,55)	16,13
12	$\overline{0}_4$	3	$(\overline{2}_4\overline{2}_4)$	$(\overline{0}_4\overline{0}_4)$	(78,66)	19,16
					(48,24)	12,6

Table 2(a): x,y class structure ($N = 1,2,...,12$)

N	Class of N	Row of N	Class of (x,y)	Class of (x',y')	$(x,y)/(x',y')$	Rows of (x,y)
13	$\bar{1}_4$	3	$(\bar{3}_4\bar{2}_4)$		(91,78)	22,19
14	$\bar{2}_4$	3	$(\bar{1}_4\bar{3}_4)$	$(\bar{3}_4\bar{3}_4)$	(105,91)	26,22
					(63,35)	15,8
15	$\bar{3}_4$	3	$(\bar{0}_4\bar{1}_4)$		(120,105)	30,26
16	$\bar{0}_4$	4	$(\bar{0}_4\bar{0}_4)$	$(\bar{0}_4\bar{0}_4)$	(136,120)	34,30
					(80,48)	20,12
17	$\bar{1}_4$	4	$(\bar{1}_4\bar{0}_4)$		(153,136)	38,34
18	$\bar{2}_4$	4	$(\bar{3}_4\bar{1}_4)$	$(\bar{3}_4\bar{3}_4)$	(171,153)	42,38
					(99,63)	24,15
19	$\bar{3}_4$	4	$(\bar{2}_4\bar{3}_4)$		(190,171)	47,42
20	$\bar{0}_4$	5	$(\bar{2}_4\bar{2}_4)$	$(\bar{0}_4\bar{0}_4)$	(210,190)	52,47
					(120,80)	30,20
21	$\bar{1}_4$	5	$(\bar{3}_4\bar{2}_4)$		(231,210)	57,52
22	$\bar{2}_4$	5	$(\bar{1}_4\bar{3}_4)$	$(\bar{3}_4\bar{3}_4)$	(253,231)	63,57
					(143,99)	35,24
23	$\bar{3}_4$	5	$(\bar{0}_4\bar{1}_4)$		(276,253)	69,63
24	$\bar{0}_4$	6	$(\bar{0}_4\bar{0}_4)$	$(\bar{0}_4\bar{0}_4)$	(300,276)	75,69
					(168,120)	42,30
25	$\bar{1}_4$	6	$(\bar{1}_4\bar{0}_4)$		(325,300)	81,75

Table 2(b): x,y class structure (N = 13,14,...25)

From Table 1 the parities of the various components of Equation (1.5), for the various class systems can be listed as in Table 3.

System	X_c^2	Y_b^2	Y_a^2	X_a^2	Y_c^2	X_b^2
$\bar{1}_4\bar{0}_4\bar{1}_4$	odd	even	even	odd	even	even
$\bar{1}_4\bar{2}_4\bar{1}_4$	odd	even	even	odd	even	even
$\bar{0}_4\bar{1}_4\bar{3}_4$	even	even	odd	even	even	odd
$\bar{2}_4\bar{1}_4\bar{3}_4$	even	even	odd	even	even	odd

Table 3: Parities of the components of Equation (1.5)

Obviously, the parity structure of "odd-even-even" is not consistent with the results from the pPts. Division by 4 can yield two odd components but the third component will then be non-integer. Hence, we need only consider the STEs obtained from Equation (1.10) and analyse why Equation (1.5) is incompatible with these. Examples of Equation (1.10) derived

STEs are displayed in Table 4.

Class types for X_zY_z	Component values	Values of rows containing components
$(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)$	$(4,12,7)(8,8,9)$	$(1,3,1)(2,2,2)$
	$(8,16,11)(4,20,5)$	$(2,4,2)(1,5,1)$
	$(12,28,15)(24,24,1)$	$(3,7,3)(6,6,0)$
	$(20,36,15)(24,16,33)$	$(5,9,3)(6,4,8)$
	$(40,12,23)(28,20,33)$	$(10,3,5)(7,5,8)$
	$(56,44,47)(16,80,25)$	$(14,11,11)(4,20,6)$
$(\bar{0}_4\bar{2}_4\bar{1}_4)(\bar{2}_4\bar{0}_4\bar{3}_4)$	$(4,6,5)(2,8,3)$	$(1,1,1)(0,2,0)$
	$(4,22,5)(10,8,19)$	$(1,5,1)(2,2,4)$
	$(12,22,13)(10,24,11)$	$(3,5,3)(2,6,2)$
$(\bar{1}_4\bar{0}_4\bar{0}_4)(\bar{3}_4\bar{2}_4\bar{2}_4)$	$(13,12,4)(15,10,2)$	$(3,3,1)(3,2,0)$
	$(17,24,8)(11,18,22)$	$(4,6,2)(2,4,5)$
$(\bar{3}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{2}_4\bar{2}_4)$	$(3,4,4)(1,6,2)$	$(0,1,1)(0,1,0)$
	$(19,28,32)(25,38,10)$	$(4,7,8)(6,9,2)$
$(\bar{3}_4\bar{2}_4\bar{2}_4)(\bar{1}_4\bar{2}_4\bar{2}_4)$	$(27,30,2)(21,34,6)$	$(6,7,0)(5,8,1)$
$(\bar{3}_4\bar{2}_4\bar{2}_4)(\bar{3}_4\bar{0}_4\bar{0}_4)$	$(11,14,14)(7,20,8)$	$(2,3,3)(1,5,2)$
$(\bar{1}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{2}_4\bar{2}_4)$	$(5,4,12)(9,10,2)$	$(1,1,3)(2,2,0)$
$(\bar{1}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{0}_4\bar{0}_4)$	$(5,4,16)(13,8,8)$	$(1,1,4)(3,2,2)$

Table 4: STEs from Equation (1.10)

In Table 5 we list the possible $\mathbb{Z}_4$ systems for $(X_cY_bY_a)$ and $(X_aY_cX_b)$ that are permissible for c,b,a.

Class of n	c,b,a Classes	$(X_cY_bY_a)(X_aY_cX_b)$
$\bar{1}_4 \vee \bar{3}_4$	$(\bar{1}_4\bar{0}_4\bar{1}_4) \vee (\bar{1}_4\bar{2}_4\bar{1}_4)$	$(\bar{1}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{0}_4\bar{0}_4)\ *$
		$(\bar{1}_4\bar{0}_4\bar{2}_4)(\bar{3}_4\bar{0}_4\bar{0}_4)$
		$(\bar{3}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{2}_4\bar{0}_4)$
		$(\bar{3}_4\bar{0}_4\bar{2}_4)(\bar{3}_4\bar{2}_4\bar{0}_4)\ *$
$\bar{3}_4$	$(\bar{0}_4\bar{1}_4\bar{3}_4)$	$(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{2}_4\bar{0}_4\bar{1}_4)$
		$(\bar{0}_4\bar{0}_4\bar{1}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)\ *$
		$(\bar{0}_4\bar{2}_4\bar{3}_4)(\bar{2}_4\bar{0}_4\bar{3}_4)\ *$
		$(\bar{0}_4\bar{2}_4\bar{1}_4)(\bar{0}_4\bar{0}_4\bar{3}_4)$

TaBle 5(a): Possible $\mathbb{Z}_4$ systems for $(X_cY_bY_a)$ and $(X_aY_cX_b)$

Class of n	c,b,a Classes	$(X_cY_bY_a)(X_aY_cX_b)$
$\bar{1}_4$	$(\bar{0}_4\bar{1}_4\bar{3}_4)$	$(\bar{0}_4\bar{0}_4\bar{1}_4)(\bar{2}_4\bar{0}_4\bar{1}_4)$
		$(\bar{0}_4\bar{2}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{3}_4)$
		$(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)$ *
		$(\bar{0}_4\bar{2}_4\bar{1}_4)(\bar{2}_4\bar{0}_4\bar{3}_4)$ *
$\bar{3}_4$	$(\bar{2}_4\bar{1}_4\bar{3}_4)$	$(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{2}_4\bar{0}_4\bar{1}_4)$
		$(\bar{0}_4\bar{0}_4\bar{1}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)$ *
		$(\bar{0}_4\bar{2}_4\bar{3}_4)(\bar{2}_4\bar{0}_4\bar{3}_4)$ *
		$(\bar{0}_4\bar{2}_4\bar{1}_4)(\bar{0}_4\bar{0}_4\bar{3}_4)$
$\bar{1}_4$	$(\bar{2}_4\bar{1}_4\bar{3}_4)$	$(\bar{0}_4\bar{0}_4\bar{1}_4)(\bar{2}_4\bar{0}_4\bar{1}_4)$
		$(\bar{0}_4\bar{2}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{3}_4)$
		$(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)$ *
		$(\bar{0}_4\bar{2}_4\bar{1}_4)(\bar{2}_4\bar{0}_4\bar{3}_4)$ *

Table 5(b): Possible $\mathbb{Z}_4$ systems for $(X_cY_bY_a)$ and $(X_aY_cX_b)$

Non-asterisked sets are incompatible with each other. This is because squares of integers from $\bar{2}_4$ are in odd rows of $\bar{0}_4$, whereas squares of integers from $\bar{0}_4$ are in even rows in $\bar{0}_4$. The odd squares are always in $\bar{1}_4$ in an even row, r_1, with $6|r_1$, unless $3|N$ when

$$r_j = 2 + 18\sum_{i=1}^{j} i$$

with

$$r_1 = 6K_i,\ i = 0,1,2,3,\ldots$$

and K_i is a generalized pentagonal number defined by

$$K_i = \begin{cases} \tfrac{1}{2}n(3n-1), & \text{even } i, \\ \tfrac{1}{2}n(3n+1), & \text{odd } i, \end{cases}$$

with $n = 1,2,3,\ldots$ (adaptations of Honsberger (1970) and Niven and Zuckerman (1966)). This gives

$$n = \begin{cases} \tfrac{1}{2}(i+2), & i \text{ even}, \\ \tfrac{1}{2}(i+1), & i \text{ odd}. \end{cases}$$

Hence, for the system $(\bar{1}_4\bar{0}_4\bar{2}_4)(\bar{3}_4\bar{0}_4\bar{0}_4)$ we obtain for the X_z, Y_z squares the Class systems:

$$(\bar{1}_4\bar{0}_4\bar{0}_4) = (\bar{1}_4\bar{0}_4\bar{0}_4),$$

which, in terms of the row functions, becomes

$$(4R_1+1)+4R_0+4R_0' = (4R_1'+1)+4r_0+4r_0'. \qquad (2.1)$$

R_0' will be odd (as $Y_a \in \bar{2}_4$), but all other rows will be even so there is parity incompatibility.

We know from Fermat's Last Theorem that even the asterisked sets cannot give integer solutions for all the components. Why does the integer structure prevent this? Here we illustrate the answer to this question for a few representative sets. As shown previously (Leyendekkers and Shannon, 1999,2000) an exclusion row factor underlies the failure to find integer solutions; that is, the class and row structures of the systems cannot accommodate a sum or difference of two same-powers in a site reserved for that power in $\mathbb{Z}_4$ when $n > 2$.

Consider the system (c,b,a) $(\bar{2}_1\bar{1}_4\bar{3}_4) \in \bar{1}_4$ with power $n \in \{5,9,13,...\}$ and the $(X_cY_bY_a)(X_aY_cX_b)$ systems $(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)$ (Table 5). We take $n = 9$ as an example. Then with

$$X_z = z^{½(n-3)}x_z = z^3x_z$$

and

$$Y_z = z^3y_z,$$

the following relationships can be deduced.

<u>Consider component c.</u>

Here

$$c = 4r_2+2,$$
$$c^3 = (4r_2+2)^3 = 4R_0,$$

(Class $\bar{2}_4$ contains no powers), where

$$R_0 = 16r_2^3+24r_2^2+12r_2+2$$

and must be even, and (x_c,y_c) are given in Table 6. Since $c^3 \in \bar{0}_4$ and $\bar{0}_4 \times \bar{A}_4 = \bar{0}_4$ (where A represents any of the four classes), three possible (x_c,y_c) need to be considered (Tables 6,7).

Class of z	x_z	y_z	Parity of row of z	Parity of rows of x_z,y_z & x_z',y_z'
$\bar{1}_4$	$\bar{1}_4$	$\bar{0}_4$	e	same
	$\bar{3}_4$	$\bar{2}_4$	o	opposite
$\bar{3}_4$	$\bar{2}_4$	$\bar{3}_4$	e	opposite
	$\bar{0}_4$	$\bar{1}_4$	o	same
$\bar{0}_4$	$\bar{2}_4$	$\bar{2}_4$	o	opposite
	$\bar{0}_4$	$\bar{0}_4$	e	same (even)*
$\bar{2}_4$	$\bar{1}_4$	$\bar{3}_4$	o	same
	$\bar{3}_4$	$\bar{3}_4$	o or e	opposite*
	$\bar{3}_4$	$\bar{1}_4$	e	same

Table 6: Row parity constraints from Table 2;

* parity of rows of x'_z, y'_z

As can be seen (Table 7), X_c and Y_c always fall in rows of the same parity, namely even. This result applies for all $n \in \bar{1}_4, n > 5$, since

$$(4r_2+2)^m = (4r_2)^m + 2m(4r_2)^{m-1} + \ldots + 2^m \qquad (2.2.)$$

with $m = \frac{1}{2}(n-3)$, $\frac{1}{4}2^m$ will always be even $(n > 5, n \in \{9, 13, 17, \ldots\})$. Hence, R_0 will always be even.

x_c	y_c	X_c	Y_c	Parity of rows of X_c, Y_c
$4r_1+1$	$4r_3+3$	$4(4r_1R_0+R_0)$	$4(4r_3R_0+3R_0)$	same (even)
$4r_3+3$	$4r'_3+3$	$4(4r_3R_0+3R_0)$	$4(4r'_3R_0+3R_0)$	same (even)
$4r_1+3$	$4r_1+1$	$4(4r_3R_0+3R_0)$	$4(4r_1R_0+R_0)$	both even

Table 7: Row parity constraints for X_c, Y_c

Consider component b.
Here

$$b = 4r_1+1,$$
$$b^3 = 4R'_1+1$$

with

$$R'_1 = 16r_1^3 + 12r_1^2 + 3r_1,$$

and from Table 6 for the (x_z, y_z) triple system of $(\bar{0}_4\bar{0}_4\bar{3}_4)(\bar{0}_4\bar{0}_4\bar{1}_4)$, b will be in an even row, that is, r_1 is even, and so R'_1 is also even. With $x_b \in \bar{1}_4, y_b \in \bar{0}_4$, and with $X_b \in \bar{1}_4, Y_b \in \bar{0}_4$,

$$X_b = 4(4R_1R'_1 + R'_1 + R_1) + 1, \qquad (2.3)$$
$$Y_b = 4(4R'_1r_0 + r_0), \qquad (2.4)$$

in which R_1 and r_0 represent the rows of x_b and y_b, respectively. Since, R_1 and r_0 have the same parity, we can construct Table 8, where it can be seen that X_b and Y_b fall in rows with the same parity..

R'_1	R_1	row of X_b	R'_1	r_0	row of Y_b
even	odd	odd	even	odd	odd
even	even	even	even	even	even

Table 8: Parities - component b

Consider component a.

Here

$$a = 4r_3 + 3,$$
$$a^3 = 4R_3 + 3,$$

with

$$R_3 = 16r_3^3 + 36r_3^2 + 27r_3 + 6.$$

Since $x_a \in \overline{0}_4$ and $y_a \in \overline{1}_4$, r_3 is odd (Table 6) and so R_3 is odd. With $x_a = 4r_0$ and $y_a = 4r_1 + 1$,

$$X_a = 4(4R_3r_0 + 3r_0), \qquad (2.5)$$
$$Y_a = 4(4R_3r_1 + 3r_1 + R_3) + 3. \qquad (2.6)$$

From Table 6, r_0 and r_1 have the same parity so that Table 9 can be constructed. This component differs from the other two in that the rows in which X_a and Y_a fall are of opposite parity.

R_3	r_0	Row of X_a	R_3	r_1	Row of Y_a
odd	even	even	odd	even	odd
odd	odd	odd	odd	odd	even

Table 9: Parities - component a

These results show that the possible row combinations for the (X_z, Y_z) system $(\overline{0}_4\overline{0}_4\overline{3}_4)(\overline{0}_4\overline{0}_4\overline{1}_4)$ are those in Table 10.

X_c	Y_b	Y_a	X_a	Y_c	X_b
even	even	odd	even	even	even
even	odd	even	odd	even	odd

Table 10: Permissible rows for the X_z, Y_z components

As can be seen from Table 4, the row systems for the (X_zY_z) system $(\overline{0}_4\overline{0}_4\overline{3}_4)$ $(\overline{0}_4\overline{0}_4\overline{1}_4)$ are $(o,o,o)(e,e,e)$; $(e,e,e)(o,o,o)$, $(e,o,o)(o,o,e)$ or $(e,o,o)(e,e,e)$ which do not appear in Table 10. Similar row systems operate for the (X_zY_z) system $(\overline{0}_4\overline{2}_4\overline{1}_4)(\overline{2}_4\overline{0}_4\overline{3}_4)$ so that viable systems are ruled out for $(c,b,a)(\overline{2}_4\overline{1}_4\overline{3}_4)$ with $n = \overline{1}_4$ (Table 5).

The (c,b,a) system $(\overline{0}_4\overline{1}_4\overline{3}_4)$ with $n \in \overline{1}_4$ also contains these (X_zY_z) systems. Here $c = 4r_0, b = 4r_1 + 1, a = 4r_3 + 3$ so that for $n = 9$

$$X_c = (4r_0)^3 x_c = 4R_0 x_c \qquad (2.7)$$

with $x_c \in \{\overline{2}_4, \overline{0}_4\}$ and R_0 is even.

$$Y_c = 4R_0 y_c \qquad (2.8)$$

with $y_c \in \{\overline{2}_4, \overline{0}_4\}$.

Thus, both rows for x_c and y_c will be even, whereas they should be of opposite parity if they are to fit the $(\overline{0}_4\overline{0}_4\overline{3}_4)$ $(\overline{0}_4\overline{0}_4\overline{1}_4)$ or $(\overline{0}_4\overline{2}_4\overline{1}_4)(\overline{2}_4\overline{0}_4\overline{3}_4)$ row systems. Similar analyses can be made for the other systems in Table 5.

Even n.

When n is even, the various solutions may be found in Hunter (1964). The class structures of (c, b, a) permissible for Equation (1.4) are $(\overline{1}_4, \overline{0}_4, \overline{1}_4)$, $(\overline{1}_4, \overline{2}_4, \overline{1}_4)$, $(\overline{1}_4, \overline{0}_4, \overline{3}_4)$, $(\overline{1}_4, \overline{2}_4, \overline{3}_4)$. In this case, $z^{½(n-3)}$ gives a fractional power so that we use X_z^2, Y_z^2 in Table 11, in which $X_z^2 = z^{n-3}x_z^2$ and $Y_z^2 = z^{n-3}y_z^2$.

As can be seen from Table 11, there are six even components and two odd (X_z, Y_z) so that the same result applies as for odd n when $n > 2$. That is, STEs from pPts can never be formed from (X_z, Y_z) systems. Furthermore, when $z \in \overline{3}_4$ we get the anomalous result that $Y_z^2 \in \overline{3}_4$ (Table 11), which is impossible since $\overline{3}_4$ contains no squares. This situation arises because $X_z^2 - Y_z^2 = \overline{0}_4 - \overline{3}_4 = \overline{1}_4$ which is the correct class for even powers of odd components.

Class of z	n	z^{n-3}	x_z^2	y_z^2	X_z^2	Y_z^2
$\overline{1}_4$	$\overline{0}_4 \vee \overline{2}_4$	$\overline{1}_4$	$\overline{1}_4$	$\overline{0}_4$	$\overline{1}_4 \times \overline{1}_4 = \overline{1}_4$	$\overline{1}_4 \times \overline{0}_4 = \overline{0}_4$
$\overline{3}_4$	$\overline{0}_4 \vee \overline{2}_4$	$\overline{3}_4$	$\overline{0}_4$	$\overline{1}_4$	$\overline{3}_4 \times \overline{0}_4 = \overline{0}_4$	$\overline{3}_4 \times \overline{1}_4 = \overline{3}_4$
$\overline{0}_4$	$\overline{0}_4 \vee \overline{2}_4$	$\overline{0}_4$	$\overline{0}_4$	$\overline{0}_4$	$\overline{0}_4 \times \overline{0}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{0}_4 = \overline{0}_4$
$\overline{2}_4$	$\overline{0}_4 \vee \overline{2}_4$	$\overline{0}_4$	$\overline{1}_4$	$\overline{1}_4$	$\overline{0}_4 \times \overline{1}_4 = \overline{0}_4$	$\overline{0}_4 \times \overline{1}_4 = \overline{0}_4$

Table 11: Classes associated with squares

Since n is even, $z^{½(n-3)}$ and hence X_z, Y_z will only be integer if z is a square, and squares are confined to Classes $\overline{1}_4$ or $\overline{0}_4$. Thus only the (c, b, a) system $(\overline{1}_4\overline{0}_4\overline{1}_4)$ needs to be considered for the STEs of Equation (1.10). Let

$$c = d^2, b = e^2, a = f^2,$$

then

$$X_c = d^{n-3}x_c, Y_c = d^{n-3}y_c.$$

Our aim here is to show how the Class and Row structure of the integers within $\mathbb{Z}_4$ underlies the failure to find integer solutions. We now consider an example to illustrate the row exclusion factor explored previously (Leyendekkers and Shannon, 1999, 2002).

With $c \in \overline{1}_4, b \in \overline{0}_4, a \in \overline{1}_4$ consider the (X_z, Y_z) system $(\overline{3}_4\overline{0}_4\overline{0}_4)(\overline{1}_4\overline{0}_4\overline{0}_4)$. This will have $d \in \overline{3}_4$ and $f \in \overline{1}_4$; ($d^2 \in \overline{1}_4$, as there are no even powers in $\overline{3}_4$).

Consider component c.

As shown previously (Leyendekkers and Shannon, 2001): $x_c = ½c(c+1)$ and $y_c = ½c(c-1)$; $c = 4r_{1c} + 1$, so that

$$X_c = d^{n-1}(2r_{1c} + 1), \tag{2.9}$$

in which r_{1c} is the row for c in Class $\overline{1}_4$ and is always even because c is a square.

Similarly,

$$Y_c = d^{n-1}(2r_{1c}). \qquad (2.10)$$

Since $c = d^2$ and with $d = 4r_{3d} + 3$,

$$r_{1c} = 4r_{3d}^2 + 6r_{3d} + 2. \qquad (2.11)$$

Since $(n-1)$ is odd, $d^{n-1} \in \overline{3}_4$. Thus $X_c = \overline{3}_4 \times \overline{1}_4 = \overline{3}_4$; $Y_c = \overline{3}_4 \times \overline{0}_4 = \overline{0}_4$.

Since $(4r_{3d} + 3)^{n-1} = 4R_3 + 3$, and $(n-1)$ is odd, R_3 and r_{3d} will have the same parity. Therefore,

$$X_c = (4R_3 + 3)(4R_1 + 1) = 4(4R_1R_3 + 3R_1 + R_3) + 3,$$

with, from Equation (2.11),

$$R_1 = \tfrac{1}{2}r_{1c} = 2r_{3d}^2 + 3r_{3d} + 1.$$

Hence,

$$X_c = 4R_3' + 3,$$

with

$$R_3' = 4R_1R_3 + 3R_1 + R_3,$$

and we can form Table 12 in which we see that the row of X_c is always odd, independent of n.

r_{3d}	R_3	R_1	Row of X_c
odd	odd	even	odd
even	even	odd	odd

Table 12: Rows of X_c

Consider component b.

$b \in \overline{0}_4$, $b = e^2$ with $e \in \overline{0}_4$, and so $e = 4r_{0e}$, so that

$$b = 4(4r_{0e}^2) = 4R_0. \qquad (2.12)$$

Obviuosly, R_0 is always even. Moreover,

$$X_b = e^{n-3}x_b,$$
$$Y_b = e^{n-3}y_b,$$

with

$$x_b = \tfrac{1}{2}b(b+1) = 2R_0(4R_0 + 1),$$
$$y_b = \tfrac{1}{2}b(b-1) = 2R_0(4R_0 - 1).$$

Since

$$e^{n-3} = (4r_{0e})^{n-3},$$

$$X_b = (4r_{0e})^{n-3} \times 2R_0(4R_0 + 1)$$
$$= 4(4^{n-4}r_{0e}^{n-3} \times 2R_0(4R_0 + 1)). \quad (2.13)$$

Thus the row for X_b is even, and

$$Y_b = (4r_{0e})^{n-3} \times 2R_0(4R_0 - 1). \quad (2.14)$$

Clearly both rows will be even. On the other hand, when $e \in \bar{2}_4$,

$$e = 4r_{2e} + 2,$$

and

$$b = 4(4r_{2e}^2 + 4r_{2e} + 1) = 4R_0,$$

so that R_0 is odd. However, rows for X_b and Y_b will still be even.

When $(X_cY_bY_a)(X_aY_cX_b)$ fall in the class systems $(\bar{3}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{0}_4\bar{0}_4)$, in many cases, the rows for $X_cY_bY_a$ must be of the same parity and rows of $X_aY_cY_b$ must be of the same parity but opposite in parity to the $X_cY_bY_a$ set. For example $(7,12,4)(9,8,8)$ has rows $(1,3,1)(2,2,2)$ (Table 4). Other parity row sets are given in Table 4.

Since X_b and Y_b have even rows and the row which contains X_c is always odd, the required row structure for the (X_z, Y_z) components cannot be achieved, despite compliance in some cases for the row of X_c having odd parity.

Other systems, such as $(\bar{1}_4\bar{0}_4\bar{0}_4)(\bar{1}_4\bar{0}_4\bar{0}_4)$ may be explored in the same way. For some systems it might be necessary to consider the row structure of the rows themselves in order to show compatibility.

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