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Algebraic and Geometric Analyses of a Fermat/Cardano Cubic

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Abstract

It is shown that the functions $R = -x^3 + 3(p+q)x^2 - 3(p^2+q^2)x + (p^3+q^3)$, $p,q \in \mathbb{Z}_+$, and $R = 3x^2 - 3(2(p+q)-1)x + (3(p^2+q^2) - 3(p+q) + 1)$ intersect at a point that is always non-integer. A geometric analysis shows that the cubic crosses the x −axis at a point, x_0, that is always non-integer, with $x_0 = N^{\frac{1}{n}}(p+q+(2pq)^{1/2})$, $N,n \in \mathbb{Z}_+$, where $N^{\frac{1}{n}}$ is obtained from the geometry of the curve. These results show that a general parameter associated with the real roots of Fermat/Cardano polynomials is a function of p,q and the geometry of the curve, which in turn yield the link with the geometry of the complex plane.

1, Introduction

We have previously discussed a family of Cardano polynomials with no integer solutions which are related to diophantine equations which also lack integer solutions [2]. This exploited the way elementary number theory deals with arithmetic properties of the ring of integers [5]. The present paper continues the analysis of these equations by using algebraic and geometric methods to study the characteristics of the lowest member of the family, the cubic.

Essentially, we deal with corollaries to Fermat's Last Theorem (FLT) so that the results complement results obtained with the modular ring $\mathbb{Z}_4$ [1,4] whereby the integer structure itself was used to explain FLT in the context of the ring of integers having many properties in common with the ring of polynomials over a finite field [6].

2. Algebraic Analysis

Consider the equation

$$R = x^3 - (x-p)^3 - (x-q)^3 \tag{2.1}$$

$p,q \in \mathbb{Z}_+$. Plots of this cubic for a variety of (p,q) ordered pairs display the following characteristics. If we expand Equation (2.1), we obtain

$$R = -x^3 + 3(p+q)x^2 - 3(p^2+q^2)x + (p^3+q^3). \tag{2.2}$$

Differentiation of R with respect to x yields

$$\frac{dR}{dx} = -3x^2 + 6(p+q)x - 3(p^2+q^2), \tag{2.3}$$

so that

$$x = (p+q) \pm (2pq)^{½} \tag{2.4}$$

at the stationary points. R_{max}, the maximum value of R will be

$$R_{max} = 3pq(p+q) + 2(2pq)^{3/2}, \tag{2.5}$$

which is always positive. R_{min}, the minumim value of R, is given by

$$R_{min} = 3pq(p+q) - 2(2pq)^{3/2}, \tag{2.6}$$

which is also positive. The curve crosses the x –axis only once after first rising to a maximum and then curving down into the negative R quadrant. (Graphs are given in [2] where $z = -R$.) As shown previously there is only one real root when $R = 0$.

Consider the points with abscissae $x = x_1$ and $x = x_1 + 1$ on the curve $R = f(x)$, with corresponding ordinates R_1, R_2, then from Equation (2.2)

$$R_2 - R_1 = -3x_1^2 + 3(2(p+q) - 1)x_1 - 3(p^2 + q^2) + 3(p+q) - 1.. \tag{2.7}$$

If $R_2 = 0$, then

$$R_1 = 3x_1^2 - 3(2(p+q) - 1)x_1 + (3(p^2 + q^2) - 3(p+q) + 1). \tag{2.8}$$

The point of intersection of this quadratic with Equation (2.2) will give the value of x_1 and hence $(x_1 + 1)$ for $R = 0$. Equating Equations (2.2) and (2.8) gives the value of x_1 at the point of intersection as

$$x_1^3 + 3(1 - (p+q))x_1^2 - 3(2(p+q) - (p^2 + q^2) - 1)x_1 + 3(p^2 + q^2 - (p+q)) - (p^3 + q^3) + 1 = 0. \tag{2.9}$$

For example, with $(p,q) = (2,3)$

$$R_1 = -x_1^3 + 15x_1^2 - 39x_1 + 35 \tag{2.10}$$

and

$$R_1 = 3x_1^2 - 27x_1 + 25 \tag{2.11}$$

so that at the point of intersection

$$x^3 - 12x^2 + 12x - 10 = 0 \tag{2.12}$$

but this has no integer solution.

Equation (2.9) can be reduced to the Cardano form by substituting in

$$y = x_1 - (p+q-1)$$

to yield

$$y^3 - 6pqy - 3pq(p+q) = 0 \tag{2.13}$$

so that $(x_1 + 1)$, the value corresponding to $R = 0$ can be found. This value will always be non-integer according to Fermat's Last Theorem (FLT). Hence a corollary to that theorem is that a quadratic of the form of Equation (2.8) can never intersect a cubic of the form of Equation (2.2) at an integer value of x.

As shown previously [2], the cubic Cardano equation is the member of lowest degree of a family of polynomials with the same basic characteristics. Thus the point of intersection of this type of polynomial of degree n (n odd) with a related one of degree $(n-1)$ could be expected to be non-integer.

More generally, with $t \in \mathbb{Z}$ we can deduce a quadratic of the form

$$R_1 = 3tx_1^2 + 3(t^2 - 2(p+q)t)x_1 + t^3 - 3(p+q)t^2 + 3(p^2+q^2)t, \qquad (2.14)$$

and this quadratic can never intersect the Fermat/Cardano cubic above at an integer value of x. For example, with $t = 7$, Equation (2.11) becomes

$$R_1 = 21x_1^2 - 63x_1 - 119, \qquad (2.15)$$

and Equation (2.12) becomes

$$x^3 + 6x^2 - 24x - 154 = 0, \qquad (2.16)$$

for which there are no integer solutions.

3. Geometric Analysis

If one draws a triangle ABC with the base BC having B at $(x,R) = (0,0)$, C at $(x_0,0)$ and A at (x_m,R_m) with $m = \max$, then BC or $\underline{a}$ will correspond to the abscissa where the cubic crosses the x –axis, that is $R = 0$. We know from FLT that $x_0 \notin \mathbb{Z}$, so that, as a corollary, triangles of this type will always have a non-integer base: what are the other characteristics of such triangles?

A perpendicular AD to the base has a length of R_m and BD has a length of x_m. Both of these lengths are known for a given (p,q) pair (Equations (2.4) and (2.5)), and hence $\angle B$ is known.

Since $x_m \ll R_m$ (1:20 for $(p,q) = (2,3)$ and 1:345 for $(p,q) = (6,17)$), the triangle has a 'spire' shape with $\angle B$ and $\angle C$ close to 90^0. With well-known trigonometric relationships between the angles and sides of a triangle [7], it can be shown that

$$a^2 = (c^2 - b^2)\frac{\sin(C+B)}{\sin(C-B)}, \qquad (3.1)$$

but

$$c^2 = x_m^2 + R_m^2 \qquad (3.2)$$

and

$$b^2 = (\underline{a} - x_m)^2 + R_m^2, \qquad (3.3)$$

so that

$$\underline{a} = 2x_m Q, \qquad (3.4)$$

with

$$Q = 1/(1 + (\sin((C-B)/\sin(C+B))).$$

Now

$$C - B \sim 0,$$

and

$$C + B \sim 180^o,$$

so that both sines will be very small. Furthermore,

$$\sin(C + B) > \sin(C - B),$$

so that

$$0 < \frac{\sin(C - B)}{\sin(C + B)} < 1.$$

Values of the various quantities, for a range of (p, q) pairs, are listed in Table 1. $\measuredangle B$ and $\measuredangle C$ approach 90^O as p and q increase, but $C > B$ and this gives the characteristic shape of the triangle.

p,q	R_m	x_m	$\measuredangle B$	$\measuredangle C$	$\frac{\sin(C-B)}{\sin(C+B)}$
2,3	173.13844	8.4641016	87.2013	88.8330	0.4117453
1,3	65.393877	6.4494897	84.3674	87.7512	0.4304437
2,1	34	5	81.6341	86.5419	0.4175229
2,5	388.88544	11.472136	88.3103	89.3163	0.4239722
3,4	487.15	11.898979	88.6008	89.4148	0.4102368
3,11	2458.3731	22.124038	89.4844	89.7989	0.4387382
6,17	12865.406	37.282857	89.8340	89.9335	0.4283082
8,9	7128	29	89.7669	89.9022	0.4089641
24,37	312194.57	103.14261	89.9811	89.9921	0.4121703
54,7	110747.13	88.495454	89.9542	89.9839	0.4785522

Table 1: Sine ratios

p,q	$2Q$	N	n
2,3	1.4166861	12878365	47
1,3	1.3981676	13564459	49
2,1	1.410912	3784398	44
2,5	1.4045218	12060554	48
3,4	1.4182015	9548722	46
3,11	1.390107	3803808	46
6,17	1.400258	1382734	42
8,9	1.4194826	2451676	42
24,37	1.4162598	12697482	47
54,7	1.3526746	12146190	54

Table 2: Some ranges of 2Q

For the (p,q) range considered, $2Q$ ranges from 1.35... to 1.419... (Table 2) and is found empirically to be of the form $N^{1/n}$. Since rational numbers may also be expressed in this form (for example,

$$1.1 = 1104923^{1/146},$$

and

$$1.3 = 6864377^{1/60})$$

and $2Q$ values here appear to be irrational, there could be functional differences between the n and N of rational and irrational numbers. Furthermore, since

$$2Q = 1 + \frac{\tan B}{\tan C},$$

and $\tan\theta$ in the θ range of Table 1 goes from 6 to ∞, obviously $(\tan B/\tan C)$ will have a large number of rational values in the range. Of course, even rational values of $2Q$ must give non-integer values for $\underline{a}$ in Equation (3.4) in order to conform to FLT. However, the angle pairings in Table 1 are not consistent with rational values of $(\tan B/\tan C)$ in the restricted range of the 'spire' triangle.

From Equation (2.4), the value of $x_m \in \triangle ABC$ is given by

$$x_m = (p+q) + (2pq)^{\frac{1}{2}}, \tag{3.5}$$

so that the base $\underline{a} \in \triangle ABC$ can be represented as

$$\underline{a} = x_0 = N^{1/n}((p+q) + (2pq)^{\frac{1}{2}}), \tag{3.6}$$

which cannot be integer. Similar analyses can be made for the higher members of the Cardano family.

We have previously shown [4] that the roots of the Fermat/Cardano polynomials are given by

$$x_0 = (p+q) + (n-1)(2pq+e)^{\frac{1}{2}},\ n \text{ odd or even}, \tag{3.7}$$

and

$$x_0 = (p+q) - (2pq+d)^{\frac{1}{2}},\ n \text{ even}. \tag{3.8}$$

Thus, for the cubic

$$x_0 = (p+q) + 2(2pq+e)^{\frac{1}{2}}, \tag{3.9}$$

or, we can use [4]

$$x_0 = (p+q) + (E+2)(2pq)^{\frac{1}{2}}. \tag{3.10}$$

Equating Equations (3.6) and (3.10), we obtain

$$E = (N^{1/n} - 2) + \frac{(p+q)(N^{1/n} - 1)}{(2pq)^{\frac{1}{2}}}, \tag{3.11}$$

so that the general parameter E can be related to the geometry of the curve for the cubic. Furthermore [4]

$$E = \left(\frac{3}{(3 - \tan^2\theta)^{½}} - 1 \right) \qquad (3.12)$$

where θ ($52^o < \theta < 60^o$) is the angle for the complex conjugate pair of the remaining roots of the cubic. This links the geometry of the real and complex planes.

4. Concluding Comments

The same geometric analysis can be made for all higher polynomial powers, as each has only one maximum to the curve and the same type of 'spire' triangle arises. For example, in the quartic case there are only two real roots and the triangle will embrace the positive root since the maximum occurs in the positive (x,R) quadrant. This applies to all even powers with two real roots and the remainder complex. All odd powers have only one rational root and the 'spire' triangle always results. (Further research would be to apply Turner's transformations in Fibonacci geometry to these 'spire' triangles [9].) Table 3 displays some results for the quartic and quintic cases.

power	4	4	5
p,q	2,3	3,4	2,3
x_0	15.532736	21.870141	19.0803408
x_m	11.9909756	18.632807	15.532737
R_m	4175.018	14964.169	141089.3
$\hat{C}$	89.951511	89.987605	89.998559
$\hat{B}$	89.835479	89.928658	89.993692
$2Q$	1.2947356	1.1737542	1.228439
N	11679080	4779773	14069792
n	63	96	80

Table 3: Quartic and quintic cases

Churchhouse [1] has expounded the merits of using the computer to generate and test number theoretic conjectures without going so far as trying to find machine proofs of any of the results. One can thus use the computer in interactive mode to probe for evidence. For instance, in the context of this paper, some exploratory graphs for $x = N^{1/n}$ follow for irrational and rational x with Table 4 providing the key to the points highlighted.

Key	x	$N = x^n$	n
A	$\sqrt{7} - 1 = 1.6457513$	8387928	32
B	$\sqrt{10} - 2 = 1.1622777$	1861013	96
C	$\sqrt{17} - 3 = 1.1231056$	1787124	124
D	$\sqrt{18} - 3 = 1.242640$	2603147	68
E	$\sqrt{19} - 3 = 1.3588989$	6202191	51
F	$\sqrt{20} - 3 = 1.4721360$	5222961	40
G	$\sqrt{26} - 4 = 1.09900195$	2493688	156
H	$\sqrt{27} - 4 = 1.1961524$	1670790	80
I	$\sqrt{30} - 4 = 1.4772256$	4059064	39
J	$\sqrt{31} - 4 = 1.5677604$	1774129	32
K	$\sqrt{37} - 5 = 1.0827625$	3363398	189
L	$\sqrt{38} - 5 = 1.164414$	1637183	94
M	$\sqrt{39} - 5 = 1.244998$	7114477	72
N	$\sqrt{51} - 6 = 1.1414284$	1402844	107
O	$\sqrt{52} - 6 = 1.2111025$	2097731	76
P	$\sqrt{53} - 6 = 1.2801099$	4459947	62
Q	$\sqrt{54} - 6 = 1.3484692$	5645912	52
R	$\sqrt{71} - 7 = 1.4261498$	8658270	45
S	$\sqrt{74} - 7 = 1.6023253$	3564690	32

Table 4: Legend for graphs

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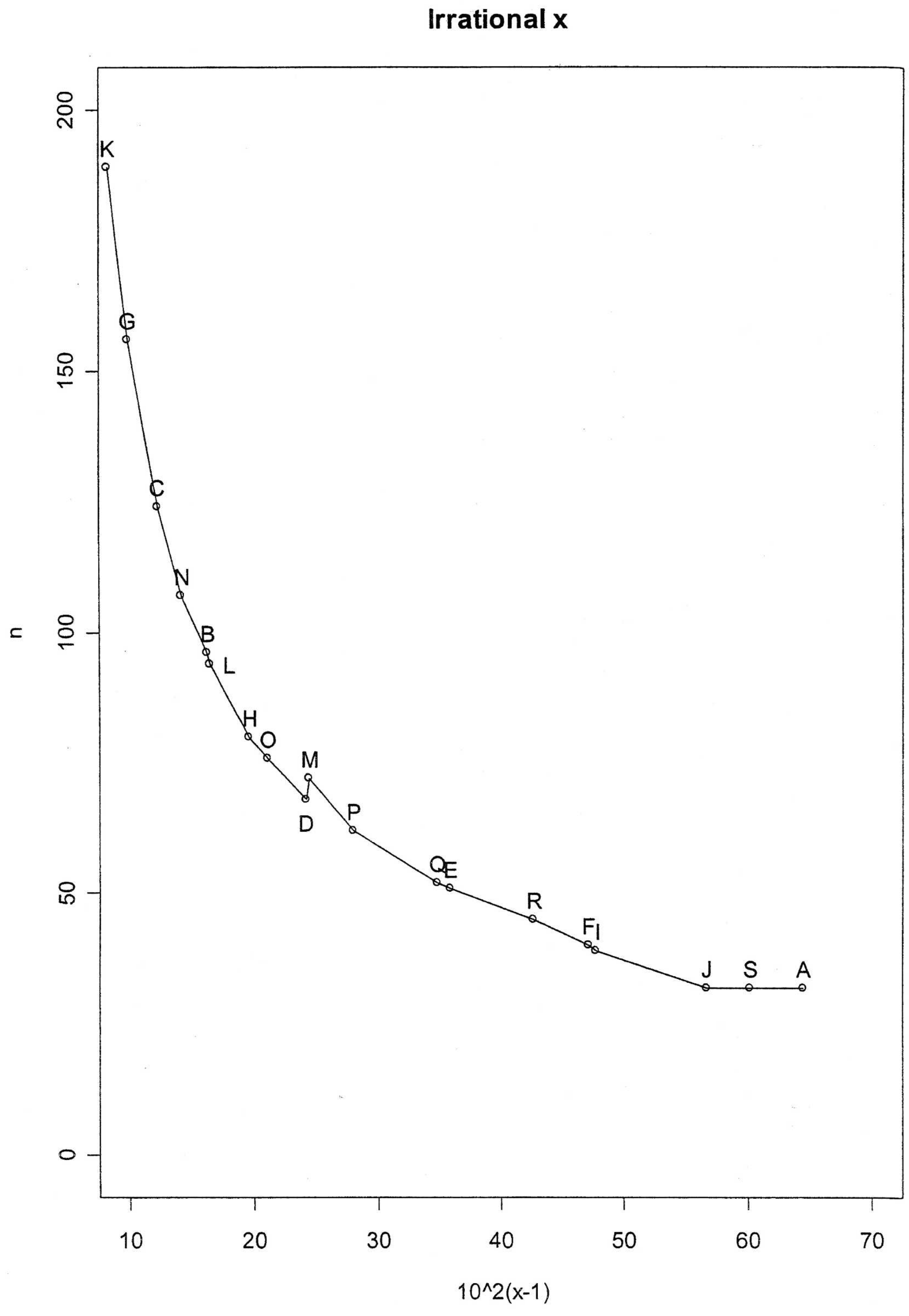

Irrational x
n
10^2(x-1)
200
150
100
50
0
10
20
30
40
50
60
70
K
G
C
N
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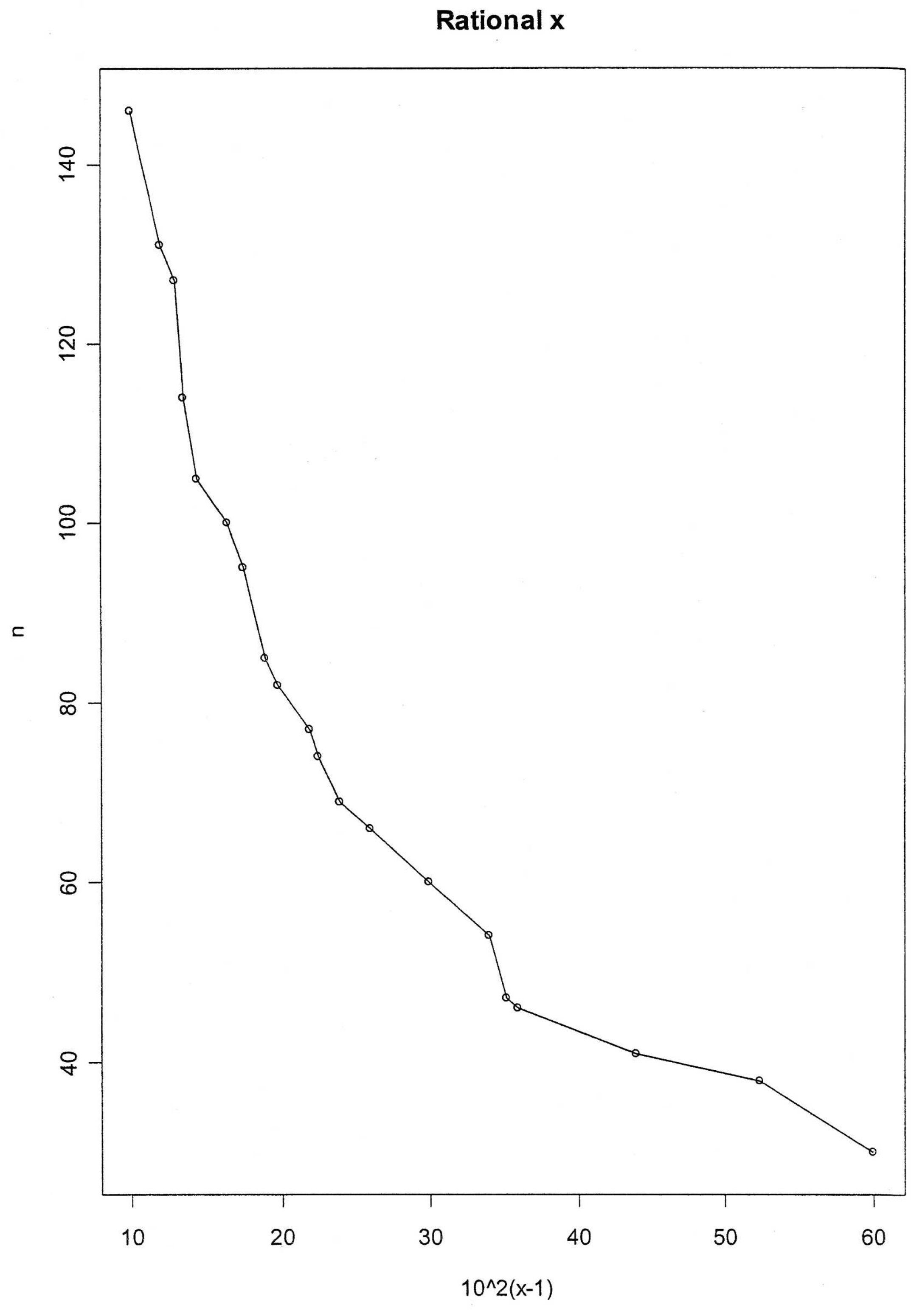

Rational x
n
140
120
100
80
60
40
10
20
30
40
50
60
10^2(x-1)

Rational and Irrational x