NNTDM 4 (2002) 3, 77-84

ABOUT TARTAGLION'S REPRESENTATION OF PLANES

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ABSTRACT: In the paper $\Re^3$ is considered, but as a special and a new commutative ring. The elements of this ring are called tartaglions. Necessary and sufficient conditions, for a plane in $\Re^3$ to be representated by Cartesian tartaglion's equation, are estabilished.

1. USED DENOTATIONS: $\Re^n$ - for the standard n - dimensional vector space over the real number field. Further the elements of $\Re^3$ are called vectors (or points, if there is no danger of missunderstanding) and are written as 3-tuples (for example: (u, v, w)); $\Im$ - for the commutative ring (with respect to matrix addition and multiplication) of all circulant matrices of order 3, over the real number field, i.e. the matrices of the kind: $\begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$, with a, b, c are real numbers; $\oplus$ - for the vector's addition; $\times$ - for the vector's multiplication; Ξ - for the set of all planes in $\Re^3$; $\Delta(u, v, w)$ - for the expression: $u^3 + v^3 + w^3 - 3uvw$, where u, v, w are arbitrary real numbers.

2. THE RING OF TARTAGLIONS (TAR)

Aceept standard vector additon $\oplus$, which is an additive operation, we introduce also the multiplicative operation: $\odot : \Re^3 \times \Re^3 \to \Re^3$, given by

$$(a, b, c) \odot (a_1, b_1, c_1) = (aa_1 + bc_1 + cb_1, ab_1 + ba_1 + cc_1, ac_1 + bb_1 + ca_1). \qquad (1)$$

The operations $\oplus$ and $\odot$ transform $\Re^3$ into a commutative ring (with the unit element $e = (1, 0, 0)$), which we denote by $\Re^3(\oplus, \odot)$.

Definition 1. We shall denote by TAR the ring $\Re^3(\oplus, \odot)$ and we shall call the elements of TAR tartaglions (in honour of the famous Italian mathematician Nicolo Tartaglia).

It is clear that the map: $\tau : TAR \to \Im$, given by

$$\tau(a, b, c) = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} \qquad (2)$$

is an algebraic isomorphism. Therefore, we may assume that $\Im$ gives the matrix representation of TAR. Moreover, the general algebraic equation of power 3:

$$x^3 + px^2 + qx + r = 0 \qquad (3)$$

could be solved easily if we consider it as a characteristic equation of the matrix from (2), i.e. like the equation

$$det \begin{pmatrix} a-x & c & b \\ b & a-x & c \\ c & b & a-x \end{pmatrix} = 0, \tag{4}$$

where the notation *det* is used for the determinant of the matrix from (4).

Now it is clear why the name of Tartaglia was used in Definition 1, since Tartaglia was one of the first who solved the equation (3).

Definition 2. We introduce TAR_0 as the subset of TAR, such that $(a,b,c) \in TAR_0$ iff $(a,b,c) \in TAR$ & $c = 0$.

We must note, that if the variables: $x, y, z \in \Re^1$ (independently of one another), then the tartaglion's variable $Y = (x,y,z) \in TAR$. Also, if the variables: u, v run $\Re^1$ (independently of one another), then the tartaglion's variable $X = (u,v,0) \in TAR_0$.

3. CARTESIAN TARTAGLION'S EQUATION OF A PLANE

Let $K = (a,b,c) \in TAR$ and $B = (d,e,f) \in TAR$ are fixed. Then the tartaglion's equation

$$Y = (K \odot X) \oplus B \tag{5}$$

may represent a plane in $\Re^3$. Below we shall discuss this question, but firstly we note that (5) looks like the famous straight-line equation

$$y = (k.x) + b \tag{6}$$

in $\Re^2$ (where for (6) the Cartesian coordinates are used). For this reason, we call (5) Cartesian tartaglion's equation of a plane.

It is well known (see [1]) that an arbitrary plane in $\Re^3$ has the equation

$$\alpha_0 x + \beta_0 y + \gamma_0 z + \delta_0 = 0, \tag{7}$$

where: $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \Re^1$ are fixed and $(\alpha_0, \beta_0, \gamma_0)$ is a nonzero vector. Moreover, the equation (7) represents a plane in $\Re^3$ iff $(\alpha_0, \beta_0, \gamma_0)$ is a nonzero vector.

Of course, the variables: x, y, z from (7) admit the representation:

$$\begin{aligned} x &= p_1\lambda + q_1\mu + r_1; \\ y &= p_2\lambda + q_2\mu + r_2; \\ z &= p_3\lambda + q_3\mu + r_3, \end{aligned} \tag{8}$$

where $p_1, q_1, r_1 \in \Re^3$ $(i = 1, 2, ..., 6)$ are fixed and $(\lambda, \mu) \in \Re^2$.

We note that not only all planes in $\Re^3$, but all straight lines too, are obtained by (8), when the vectors $(p_1, p_2, p_3), (q_1, q_2, q_3)$ are not linearly independent. Also all points in $\Re^3$ are obtained by (8), when

$$(p_1, p_2, p_3) = (q_1, q_2, q_3) = (0, 0, 0).$$

Returning to (5), from the definiton of $\oplus$ and $\otimes$, we conclude that it is equivalent to:

$$\begin{aligned} x &= au + cv + d; \\ y &= bu + av + e; \\ z &= cu + bv + f \end{aligned} \tag{9}$$

Of course, (9) is a particular case from (8).

Lemma 1. The tartaglion's equation (5) represents a plane in $\Re^3$ iff the vectors: (a, b, c), (c, a, b) are linearly independent.

For the following assertion we omit the proof, since it is obvious.

Lemma 2. The vectors $(a, b, c), (c, a, b)$ are not linearly independent iff $a = b = c$.

Proof of lemma 1. Let the vectors $(a, b, c), (c, a, b)$ be linearly independent. We introduce the nonzero vector $(\alpha_0, \beta_0, \gamma_0)$ by

$$(\alpha_0, \beta_0, \gamma_0) = (a, b, c) \times (c, a, b). \tag{10}$$

Let the tartaglion $Y = (x, y, z)$ satisfy (5). We shall show that if the tartaglion $X = (u, v, 0) \in TAR_0$ then Y is in a plane in $\Re^3$.

Indeed, let $X \in TAR_0$. Since (5) is equivalent to (9), we multiply the first equality of (9) by α_0, the second by β_0, the third by γ_0 and add the three new equalities. Hence:

$$\alpha_0 x + \beta_0 y + \gamma_0 z = \alpha_0 d + \beta_0 e + \gamma_0 f, \tag{11}$$

since:

$$\alpha_0 a + \beta_0 b + \gamma_0 c = 0 \alpha_0 c + \beta_0 a + \gamma_0 b = 0,$$

because of (10). Finally, we define $\delta_0 \in \Re^1$ by

$$\delta_0 = -(\alpha_0 d + \beta_0 e + \gamma_0 f). \tag{12}$$

Then (11) yields (7), with the nonzero vector $(\alpha_0, \beta_0, \gamma_0)$ from (10). This proves that Y is in the plane given by the equation (7), with δ_0 is given by (12).

Conversely, let the tartaglion's equation (5) represents a plane in $\Re^3$. Therefore, (9) represents the same plane, since (5) and (9) are equivalent. In this case we shall prove that the vectors: $(a, b, c), (c, a, b)$ are linearly independent.

Indeed, if the opposite holds, then $a = b = c$, because of lemma 2.

Hence (9) is a straight-line's parametric representation in $\Re^3$ if $a = b = c = \rho$ and $\rho \neq 0$. If $\rho = 0$, then (9) represents a point in $\Re^3$.

Therefore, we have a contradiction to our assumption that (5), i.e. (9) represents a plane in $\Re^3$.

The lemma is proved.

Definition 3. A plane in $\Re^3$ is called TAR-attainable iff it is representable by the equation (5).

Below we shall discuss the question when a plane in $\Re^3$ is TAR-attainable.

4. THE TAR-ATTAINABILITY PROBLEM

Further we shall denote by $TAR(\Xi)$ the set of all TAR-attainable planes in $\Re^3$.

Definition 4. A nonzero vector (α, β, γ) is called attainable iff there exists a vector (a, b, c) and a real nonzero number λ, such that the representation

$$(\alpha, \beta, \gamma) = \lambda.(a, b, c) \times (c, a, b) \tag{13}$$

holds. The set of all attainable vectors is denoted by ATT.

Theorem 1. Let $\varepsilon \in \Xi$ is given by the equation

$$\alpha x + \beta y + \gamma z + \delta = 0 \tag{14}$$

Then $\varepsilon \in TAR(\Xi)$ iff $(\alpha, \beta, \gamma) \in ATT$.

Proof of theorem 1. Let $\varepsilon \in TAR(\Xi)$. Then there exist tartaglions: $K = (a, b, c)$; $B = (d, e, f)$, for which (5) is satisfied. Hence the vectors $(a, b, c), (c, a, b)$ are linearly independent, since lemma 1 holds. Just as in the proof of lemma 1, we deduce that ε satisfies the equation (7), with $(\alpha_0, \beta_0, \gamma_0)$ given by (10), and δ_0 given by (12). On the other hand ε satisfies (14). Hence (see [1]):

$$\alpha = \lambda\alpha_0, \beta = \lambda\beta_0, \gamma = \lambda\gamma_0, \delta = \lambda\delta_0,$$

where λ is a nonzero real number. Therefore $(\alpha, \beta, \gamma) = \lambda.(\alpha_0, \beta_0, \gamma_0)$. which means that $(\alpha, \beta, \gamma) \in ATT$.

Conversely, let the vector $(\alpha, \beta, \gamma) \in ATT$. Then there exists a vector (a, b, c) and a nonzero real number λ, such that the representation:

$$(\alpha, \beta, \gamma) = \lambda.(a, b, c) \times (c, a, b),$$

holds. Therefore, the vectors $(a, b, c), (c, a, b)$ are linearly independent, since (α, β, γ) is a nonzero vector. Also the relations:

$$\alpha a + \beta b + \gamma c = 0$$

$$\alpha c + \beta a + \gamma b = 0,$$

hold, since the vector (α, β, γ) is orthogonal to each one of the vectors: $(a, b, c), (c, a, b)$. The last two equalities mean that (a, b, c) and (c, a, b) are coplanar vectors with ε. Let the point $(d, e, f) \in \varepsilon$ be arbitrary. Then a point $(x, y, z) \in \varepsilon$ iff :

$(x-d, y-e, z-f), (a, b, c), (c, a, b)$ are coplanar vectors, i.e. when there exist $u, v \in \Re^1$, such that:

$$(x - d, y - e, z - f) = u.(a, b, c) + v.(c, a, b).$$

The last equality is only another form of (9). But (9) is equivalent to (5). Therefore $\varepsilon \in TAR(\Xi)$.

The theorem is proved.

The following statement is necessary for our considerations.

Lemma 3. The equality $\Delta(u,v,w)=0$ holds iff at least one of the relations: a) $u+v+w=0$; b) $u=v=w$, hold.

The proof follows immediately from the easily checked identity:

$$u^3+v^3+w^3-3uvw=\frac{1}{2}(u+v+w).((u-v)^2+(v-w)^2+(w-u)^2).$$

Remark. It is easy to check that $\Delta(u,v,w)=det_T(u,v,w)$ (see(2)).

It was shown (see theorem 1) that the TAR-attainability problem, when a plane satisfies (14), reduces to the question when the vector (α,β,γ) is attainable. To give the answer to this question we need some definitions and results.

Lemma 4. If $\Delta(\alpha,\beta,\gamma)\neq 0$, then $(\alpha,\beta,\gamma)\in ATT$.

Proof of lemma 4. Let $\Delta(\alpha,\beta,\gamma)\neq 0$. We define: λ,a,b,c by:

$$\lambda=\frac{1}{\Delta(\alpha,\beta,\gamma)};a=\gamma^2-\alpha\beta;b=\alpha^2-\beta\gamma;c=\beta^2-\alpha\gamma. \tag{15}$$

Then to prove that $(\alpha,\beta,\gamma)\in ATT$ it is enough to prove that (13) holds, since $\lambda\neq 0$.

We recall that:

$$(a,b,c)\times(c,a,b)=(det\begin{pmatrix} b & c\\ a & b\end{pmatrix},-det\begin{pmatrix} a & c\\ c & b\end{pmatrix},det\begin{pmatrix} a & b\\ c & a\end{pmatrix}).$$

Hence (13) is equivalent to

$$\alpha=\lambda(b^2-ac);\beta=\lambda(c^2-ab);\gamma=\lambda(a^2-bc), \tag{16}$$

or, which is the same, to the equalities:

$$\alpha\Delta(\alpha,\beta,\gamma)=(b^2-ac);\beta\Delta(\alpha,\beta,\gamma)=(c^2-ab);\gamma\Delta(\alpha,\beta,\gamma)=(a^2-bc).$$

because of (15). But one may check directly the last equalities, using (15), and the lemma is proved.

5. ISOTROPIC VECTORS AND THEIR PROPERTIES

Definition 5. We call a nonzero vector (α,β,γ) isotropic iff $\Delta(\alpha,\beta,\gamma)=0$. The set of all isotropic vectors is denoted by ISO.

Definition 6. We call a nonzero vector (α,β,γ) isotropic of the first kind iff the relation $\alpha+\beta+\gamma=0$ holds. The set of all isotropic vectors of the first kind is denoted by ISO_1.

It is obvious that $(\alpha,\beta,\gamma)\in ISO_1$ iff there exist real numbers s,t, at least one of them is not equal to zero, and such that the relations: $\alpha=s;\beta=-s-t;\gamma=t$, hold.

Definition 7. We call a nonzero vector (α,β,γ) isotropic of the second kind iff $\alpha=\beta=\gamma$. The set of all isotropic vectors of the second kind is denoted by ISO_2.

Lemma 5. It is fulfilled: $ISO_1\subset ISO;ISO_2\subset ISO$ and moreover:

$$ISO_1 \cap ISO_2 = \emptyset \& ISO_1 \cup ISO_2 = ISO,$$

where by $\emptyset$ is denoted the empty set.

The proof follows immediately from the above three definitions and from lemma 3.

Definition 8. Let (α, β, γ) be arbitrary vector. We call vectors: $(\alpha, \beta, \gamma); (\gamma, \alpha, \beta)$; (β, γ, α) cyclic triplets.

For example, column vectors or row vectors of any matrix $M \in \Im$ form a cyclic triplet.

The following propositions are connected with cyclic triplets.

Proposition 1. Let vectors i, j, k form a cyclic triplet. Then vectors $i \times j, j \times k, k \times i$ form a cyclic triplet too.

Proposition 2. If three vectors form a cyclic triplet and one of them is isotropic, then the others are isotropic too. Moreover, the isotropic vectors of a cyclic triplet have one and the same kind (first or second).

The proofs of these propositions are obvious and we omit them.

Definition 9. Let the vector $h = (\alpha, \beta, \gamma) \in ATT$ and (13) holds with a suitable real $\lambda \neq 0$ & vector (a, b, c) (according to definition 4). We call (a, b, c) the ATT-generator of (α, β, γ) and denote by $G(h)$ the set of all ATT-generators of h.

As we saw (see lemma 4 and definition 5), every vector which is not isotropic is an attainable vector. Now we are ready to answer the question when an isotropic vector is attainable.

Lemma 6. It is fulfilled: $ISO_2 \subset ATT$.

Proof of lemma 6. Let $j \in ISO_2$. Then $j = (m, m, m)$, where m is a nonzero real number. Let $\lambda = \text{sgn } m$. Then

$$j = \lambda.(\sqrt{|m|}, -\sqrt{|m|}, 0) \times (0, \sqrt{|m|}, -\sqrt{|m|}).$$

Therefore $j \in ATT$ (see definition 4) and the lemma is proved.

Corollary. It is fulfilled: $ATT \cap ISO \neq \emptyset$.

Lemma 7. Let $h \in ATT$ and $g \in G(h)$. Then $h \in ISO$ iff $g \in ISO$.

Proof of lemma 7. Let $h = (\alpha, \beta, \gamma)$ and $g = (a, b, c)$. Then there exists a real nonzero number λ, such that (16) hold (since $h \in ATT$ and $g \in G(h)$).
Hence:

$$\alpha^2 - \beta\gamma = \lambda^2 b\Delta(a, b, c); \beta^2 - \gamma\alpha = \lambda^2 c\Delta(a, b, c); \gamma^2 - \alpha\beta = \lambda^2 c\Delta(a, b, c).$$

We multiply the first of these equalities by α, the second by β, the third by γ and add the three new equalities to obtain:

$$\Delta(\alpha, \beta, \gamma) = \lambda^2 \Delta(a, b, c).(a\gamma + b\alpha + c\beta).$$

Hence:

$$\Delta(\alpha, \beta, \gamma) = \lambda^3 \Delta^2(a, b, c). \quad (17)$$

since (16) yields:

$$a\gamma + b\alpha + c\beta = \lambda\Delta(a, b, c),$$

Because of (17), we obtain: $\Delta(\alpha, \beta, \gamma) = 0$ iff $\Delta(a, b, c) = 0$.

Then the lemma is proved (see definition 5).

Lemma 8. It is fulfilled: $ISO_1 \cap ATT = \emptyset$.

Proof of lemma 8. Let us assume that $ISO_1 \cap ATT \neq \emptyset$. Let $h \in ISO_1 \cap ATT$ and $g = (a, b, c) \in G(h)$. Then $g \in ISO$, since $h \in ISO$ (see lemma 5) and lemma 7 holds. Hence: $(c, a, b) \in ISO$ and (c, a, b) is an isotropic vector of one and the same kind as g (see proposition 2).

If we assume that $g \in ISO_1$, then there exist real numbers: s, t, such that $g = (s, -s-t, t)$. Therefore $(c, a, b) = (t, s, -s-t)$. Hence:

$$g \times (c, a, b) = (s^2 + st + t^2, s^2 + st + t^2, s^2 + st + t^2,).$$

Hence:

$$h = (\lambda(s^2 + st + t^2), \lambda(s^2 + st + t^2), \lambda(s^2 + st + t^2))$$

for a suitable $\lambda \neq 0$, since $h \in ATT$ and $g \in G(h)$. Therefore $h \in ISO_2$.
But the last contradicts to $h \in ISO_1$, since $ISO_1 \cap ISO_2 = \emptyset$ (see lemma 5). Therefore $g \in ISO_2$, i.e. $g = (m, m, m)$, where $m \neq 0$. Hence:

$$h = \lambda.(a, b, c) \times (c, a, b) = \lambda.(m, m, m) \times (m, m, m) = (0, 0, 0),$$

with $\lambda \neq 0$, since $h \in ATT$ by assumption and $g \in G(h)$ (see definition 9 too).

So, we have proved that h is the zero vector, which is impossible.

Therefore, our assumption that $ISO_1 \cap ATT \neq \emptyset$ is wrong and the lemma is proved.

6. NECESSARY AND SUFFICIENT CONDITIONS RELATED TO THE *TAR*-ATTAINABILITY PROBLEM

Our previous investigation gives a key to the main results of this paper.

Theorem 2. Let h be a nonzero vector. Then h is attainable iff exactly one of the following propositions holds:

1) h is not isotropic;

2) h is isotropic of the second kind.

The assertion of this theorem follows immediately from: lemma 4, lemma 5, lemma 6, lemma 8.

One may put the above theorem in equivalent form:

Theorem 3. Let h be a nonzero vector. Then h is not attainable iff h is isotropic of the first kind.

As a corollary to theorem 1 and theorem 3 we obtain:

Theorem 4. Let ε be a plane in $\Re^3$ satisfying the equation (14). Then ε is not TAR-attainable iff the vector (α, β, γ) is isotropic of the first kind.

The equivalent form of theorem 4 is:

Theorem 5. Let ε be a plane in $\Re^3$. Then ε is not TAR-attainable iff the straight line δ in $\Re^3$ satisfying the equation: $x = y = z$ lies in ε or has no common points with ε.

As an equivalent statement to theorem 5, we obtain the main result in the paper:

Theorem 6. Let ε be a plane in $\Re^3$. Then ε is representable by Cartesian tartaglion's equation (5) iff the straight line δ in $\Re^3$ satisfying the equation: $x = y = z$ intersects ε.

The above theorem means that ε is expressible by Cartesian tartaglion's equation (5) iff the direction of ε in $\Re^3$ is not collinear with δ. The same situation with respect to $\Re^2$ holds, when a straight line admits the Cartesian equation (6) iff the directions of this line are not collinear with the axis of ordinates.

Summarizing we note that in the present paper necessary and sufficient conditions are found for a plane in $\Re^3$ to satisfy Cartesian straight-line's equation (6) (but in tartaglion's form (5)!).

7. CONCLUSION

The present research is the first step of a further investigation in the proposed direction, in particular the connection between so defined tartaglions and tertions from [2] will be discussed.

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