NNTDM 8 (2002), 2, 70-74

Basic properties of weakly multiplicative functions

Pentti Haukkanen
Department of Mathematics, Statistics and Philosophy,
FIN-33014 University of Tampere, Finland
mapehau@uta.fi

Abstract

An arithmetical function f is said to be weakly multiplicative if f is not identically zero and $f(np) = f(n)f(p)$ for all positive integers n and primes p with $(n,p) = 1$. Every multiplicative function is a weakly multiplicative function but the converse is not true. In this note we study basic properties of weakly multiplicative functions with respect to the Dirichlet convolution.

1 Introduction

An arithmetical function f is said to be multiplicative if f is not identically zero and $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. It is clear that if f is multiplicative, then $f(1) = 1$. In 1985, Subbarao (see [2]) defined an arithmetical function f to be weakly multiplicative if f is not identically zero and

$$f(np) = f(n)f(p)$$

for all positive integers n and primes p with $(n,p) = 1$. Weakly multiplicative functions are later named quasi-multiplicative functions but we prefer the term weakly multiplicative function because the term quasi-multiplicative function is also used to arithmetical functions f such that $f(1) \neq 0$ and $f(1)f(mn) = f(m)f(n)$ whenever $(m,n) = 1$ (see [4]).

It is easy to see that every multiplicative function is a weakly multiplicative function but the converse is not true. If f is any arithmetical function, then the function f_k ($k \geq 2$) defined as $f_k(n) = f(n^{1/k})$ is weakly multiplicative provided that f_k is not identically zero. (Here we assume that $f(n^{1/k}) = 0$ if $n^{1/k}$ is not a positive integer.) Let $\omega(n)$ denote the number of distinct prime divisors of n. Then ω_k is weakly multiplicative but not multiplicative.

The Dirichlet convolution of two arithmetical functions f and g is defined as

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

The function δ, defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, serves as the identity under the Dirichlet convolution. An arithmetical function f possesses a Dirichlet inverse f^{-1} if and only if $f(1) \neq 0$. The Dirichlet inverse is unique and is given recursively as

$$f^{-1}(1) = \frac{1}{f(1)}, \; f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d>1}} f(d) f^{-1}(n/d) \; (n > 1).$$

It is well known that the set of all arithmetical functions f with $f(1) \neq 0$ forms an abelian group under the Dirichlet convolution and the set of all multiplicative functions forms a subgroup of this abelian group (see [1, 3, 4]).

In this paper we study basic properties of weakly multiplicative functions with respect to the Dirichlet convolution.

2 Properties

We divide the class W of weakly multiplicative functions into two disjoint subclasses W_1 and W_2. The class W_1 consists of those weakly multiplicative functions f for which there exists a prime p such that $f(p) \neq 0$, that is,

$$W_1 = \{f \in W \mid \exists p \in \P : f(p) \neq 0\}.$$

If $f \in W_1$, then $f(1) = 1$. The class W_2 consists of those weakly multiplicative functions f for which $f(p) = 0$ for all primes p, that is,

$$W_2 = \{f \in W \mid \forall p \in \P : f(p) = 0\}.$$

It is easy to see that $f \in W_2$ if and only if $f(n) = 0$ whenever $n = mp$, where m is a positive integer, p is a prime and $(m, p) = 1$.

Theorem 2.1 *If $f, g \in W$, then $f \star g \in W$.*

Proof. Let n be a positive integer and p a prime such that $(n, p) = 1$.
Suppose firstly that $f, g \in W_1$. Then

$$(f \star g)(np) = \sum_{d|n} \sum_{e|p} f(de) g((n/d)(p/e)).$$

If $e = 1$, then $f(de) = f(d)f(e)$ (since $f(1) = 1$) and $g((n/d)(p/e)) = g(n/d)g(p/e)$ (since $(n/d, p) = 1$). Further, if $e = p$, then $f(de) = f(d)f(e)$ (since $(d, p) = 1$) and $g((n/d)(p/e)) = g(n/d)g(p/e)$ (since $g(1) = 1$). Thus

$$(f \star g)(np) = \sum_{d|n} \sum_{e|p} f(d)f(e)g(n/d)g(p/e) = (f \star g)(n)(f \star g)(p).$$

Suppose secondly that $f, g \in W_2$. Then

$$(f \star g)(np) = \sum_{ab=np} f(a)g(b).$$

Now, $p|a$ with $(a/p,p)=1$ or $p|b$ with $(b/p,p)=1$. Thus $f(a)=f(a/p)f(p)$ or $g(b)=g(b/p)g(p)$. Since $f(p)=g(p)=0$, we have $f(a)g(b)=0$ and therefore $(f\star g)(np)=0$. Further, $(f\star g)(p)=f(p)g(1)+f(1)g(p)=0$ and therefore $(f\star g)(n)(f\star g)(p)=0$. Thus

$$(f\star g)(np)=(f\star g)(n)(f\star g)(p).$$

Thus we have proved that $f\star g\in W$. □

Corollary 2.1 *If $f,g\in W_2$, then $f\star g\in W_2$.*

Proof. According to Theorem 2.1, $f\star g\in W$. Since $f(p)=g(p)=0$ for all primes p, we have $(f\star g)(p)=f(p)g(1)+f(1)g(p)=0$ for all primes p. Thus $f\star g\in W_2$. □

Remark 2.1 It is possible that $f,g\in W_1$ but $f\star g\notin W_1$. For example, if $f(n)=1$ and $g(n)=(-1)^{\omega(n)}$ for all positive integers n, then f and g are multiplicative functions such that $f(p)=1$ and $g(p)=-1$ but $(f\star g)(p)=0$ for all primes p and therefore $f,g\in W_1$ but $f\star g\notin W_1$.

Theorem 2.2 *If $f\in W$ with $f(1)\neq 0$, then $f^{-1}\in W$.*

Proof. Let p be a fixed prime. We prove that

$$f^{-1}(np)=f^{-1}(n)f^{-1}(p) \tag{2.1}$$

for all positive integers n with $(n,p)=1$.

Suppose firstly that $f\in W_1$. We proceed by induction on n to prove that (2.1) holds. If $n=1$, then (2.1) holds, since $f^{-1}(1)=1/f(1)=1$. Assume that $f^{-1}(mp)=f^{-1}(m)f^{-1}(p)$ whenever $1\le m<n$ with $(m,p)=1$. Then

$$\begin{aligned} f^{-1}(np) &= -\sum_{\substack{d|np\\ d>1}} f(d)f^{-1}(np/d) = -\sum_{\substack{a|n\\ b|p\\ ab>1}} f(ab)f^{-1}((n/a)(p/b)) \\ &= -\sum_{\substack{a|n\\ a>1}} f(a)f^{-1}((n/a)p) - \sum_{a|n} f(ap)f^{-1}(n/a). \end{aligned}$$

Now, $f^{-1}((n/a)p)=f^{-1}(n/a)f^{-1}(p)$ (by the induction assumption) and $f(ap)=f(a)f(p)$ (since $f\in W$). Thus

$$f^{-1}(np)=-f^{-1}(p)\sum_{\substack{a|n\\ a>1}} f(a)f^{-1}(n/a) - f(p)\sum_{a|n} f(a)f^{-1}(n/a).$$

Clearly,

$$\sum_{\substack{a|n\\ a>1}} f(a)f^{-1}(n/a) = -f^{-1}(n)$$

and

$$\sum_{a|n} f(a)f^{-1}(n/a)=\delta(n)=0$$

(since $n > 1$). Therefore (2.1) holds.

Suppose secondly that $f \in W_2$ with $f(1) \neq 0$. We proceed by induction on n to prove that

$$f^{-1}(np) = 0 \tag{2.2}$$

for all positive integers n with $(n,p) = 1$. Since $f(p) = 0$, we can conclude that (2.2) holds for $n = 1$. Assume that $f^{-1}(mp) = 0$ whenever $1 \leq m < n$ with $(m,p) = 1$. Then

$$\begin{aligned} f^{-1}(np) &= -\frac{1}{f(1)} \sum_{\substack{d|np \\ d>1}} f(d)f^{-1}(np/d) = -\frac{1}{f(1)} \sum_{\substack{a|n \\ b|p \\ ab>1}} f(ab)f^{-1}((n/a)(p/b)) \\ &= -\frac{1}{f(1)} \sum_{\substack{a|n \\ a>1}} f(a)f^{-1}((n/a)p) - \frac{1}{f(1)} \sum_{a|n} f(ap)f^{-1}(n/a). \end{aligned}$$

Now, $f^{-1}((n/a)p) = 0$ (by the induction assumption) and $f(ap) = 0$ (since $f \in W_2$). Thus (2.2) holds, which shows that (2.1) holds. □

Corollary 2.2 *If $f \in W_1$, then $f^{-1} \in W_1$.*

Proof. According to Theorem 2.2, $f^{-1} \in W$. There exists a prime p such that $f(p) \neq 0$. For this prime p we have $f^{-1}(p) = -f(p) \neq 0$. Thus $f^{-1} \in W_1$. □

Corollary 2.3 *If $f \in W_2$ with $f(1) \neq 0$, then $f^{-1} \in W_2$.*

Proof. According to Theorem 2.2, $f^{-1} \in W$. Since $f(p) = 0$ for all primes p, we have $f^{-1}(p) = -f(p) = 0$ for all primes p. Thus $f^{-1} \in W_2$. □

Remark 2.2 Using the above results we make the following observations:

1. The pair $(W, \star)$ is a commutative semigroup with identity.

2. The pair $(W_2, \star)$ is a commutative semigroup with identity.

3. The pair $(W^*, \star)$ is an abelian group, where W^* is the set of weakly multiplicative functions f such that $f(1) \neq 0$. Note that W^* is the set of units in W.

4. The pair $(W_2^*, \star)$ is an abelian group, where W_2^* is the set of weakly multiplicative functions f such that $f(1) \neq 0$ and $f(p) = 0$ for all primes p. Note that W_2^* is the set of units in W_2.

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory.* UTM. New York: Springer-Verlag, 1976.

[2] I. Kátai & M. V. Subbarao. "Quasi-additive and quasi-multiplicative functions with regularity properties." *Publ. Math.* **56** (2000): 43-52.

[3] P. J. McCarthy. *Introduction to Arithmetical Functions.* Universitext. New York: Springer-Verlag, 1986.

[4] R. Sivaramakrishnan. *Classical Theory of Arithmetic Functions.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. **126**. New York: Marcel Dekker, Inc., 1989.

AMS Classification Number: 11A25