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On multiplicatively bi-unitary perfect numbers

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1 Introduction

Let $\sigma(n)$ the sum of positive divisors of n,

$$\sigma(n) = \sum_{d|n} d,$$

$s(n)$ the sum of aliquet part of n, i. e. the positive divisors of n other than n itself, so that

$$s(n) = \sigma(n) - n.$$

It is well-known that a number n is called perfect if the sum of aliquot divisors of n is equal to n

$$s(n) = n,$$

or equivalently

$$\sigma(n) = 2n.$$

Perfect, amicable and sociable numbers are fixed points of the arithmetic function s and its iterates. (R. K. Guy [4], P. Erdős [2])

The Euclid-Euler theorem gives the form of even perfect numbers:

Lemma 1.1 *An even integer n is perfect iff there exist prime number p such that $n = 2^{p-1}q$, where $q = 2^p - 1$ is prime ("Mersenne prime").*

No odd perfect numbers are known.

The number n is called super-perfect if

$$\sigma(\sigma(n)) = 2n.$$

Suryanarayana and Kanold states [10], [6] the general form of even super-perfect numbers: $n = 2^{p-1}$, where $2^p - 1 = q$ is a prime (Mersenne prime).

No odd super-perfect numbers are known.

A divisor d of a natural number n is unitary divisor if $\left(d, \frac{n}{d}\right) = 1$, and n is unitary perfect if

$$\sigma^*(n) = 2n.$$

where $\sigma^*(n)$ the sum of unitary divisors of n. The notion of unitary perfect numbers introduced M. V. Subbarao and L. J. Waren in 1966 [9].

Five unitary perfect numbers are known, they are necessarilly even and its true that no unitary perfect numbers of the form $2^m s$ where s is a squarefree odd integer [3].

For positive integers a and b, let $(a, b)^{**}$ denote the greatest common unitary divisor of a and b. A divisor d of the positive integer n, bi-unitary divisor if $d\delta = n$ and $(d, \delta)^{**} = 1$.

Let $\tau^{**}(n)$ and $\sigma^{**}(n)$ denote the number and the sum of bi-unitary divisors

$$\tau^{**}(n) = \sum_{\substack{d\delta = n \\ (d,\delta)^{**} = 1}} 1$$

$$\sigma^{**}(n) = \sum_{\substack{d\delta = n \\ (d,\delta)^{**} = 1}} d$$

Ch. R. Wall introduced the concept of bi-unitary perfect numbers [12], an integer n is bi-unitary perfect number if it equals the sum of its bi-unitary divisors

$$\sigma^{**}(n) = 2n,$$

and proved that there are only three bi-unitary perfect numbers, namely 6, 60 and 90. Sándor in [7] introduced the concept of multiplicative divisor function $T(n)$ (see. [5]) and multiplicatively perfect and superperfect number and characterize them.

In [1] the author study the unitary multiplicatively perfect numbers.

In this paper we introduce the bi-unitary muliplicative divisor function, and the notion of bi-unitary multiplicative perfect, bi-unitary multiplicative superperfect and bi-unitary k-l m-perfect numbers and characterize them.

2 Main results

Definition 2.1 *Let $T^{**}(n)$ denote the product of all bi-unitary divisors of n:*

$$T^{**}(n) = \prod_{\substack{d \mid n \\ \left(d, \frac{n}{d}\right)^{**} = 1}} d.$$

Let $T^{**k}(n)$ the kth iterate of $T^{**}(n)$:

$$T^{**k}(n) = T^{**}\left(T^{**(k-1)}(n)\right), \quad k \geq 1.$$

Definition 2.2 *The number $n > 1$ is* **multiplicatively bi-unitary perfect** *(or shortly* m **bi-unitary perfect***) if*

$$T^{**}(n) = n^2,$$

multiplicatively bi-unitary super-perfect *(*m **bi-unitary super-perfect***), if*

$$T^{**}(T^{**}(n)) = n^2,$$

and **multiplicatively bi-unitary** (k, l) **perfect** *(*m **unitary** (k, l) **perfect***), if*

$$T^{**k}(n) = n^l.$$

First we prove the following result:

Lemma 2.1 *For $n \geq 1$*

$$T^{**}(n) = n^{\frac{\tau^{**}(n)}{2}}$$

where $\tau^{*}(n)$ denotes the number of bi-unitary divisors of n.

Proof If $d_1, d_2, \ldots, d_k$ are all bi-unitary divisors of n, then

$$\{d_1, d_2, \ldots, d_k\} = \left\{\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k}\right\},$$

implying that

$$d_1 d_2 \dots d_k = \frac{n}{d_1} \cdot \frac{n}{d_2} \cdots \frac{n}{d_k},$$

$$T^{**}(n) = n^{k/2}$$

where $k = \tau^{**}(n)$ denotes the number of bi-unitary divisors of n.

□

Remark

The $T^{**}(n)$ function not multiplicative and not additive function.

Theorem 2.2 *All m bi-unitary perfect numbers n, $(n > 1)$ have one of the following forms: $n = p_1^4$, $n = p_1^3$, $n = p_1^2 p_2^2$, $n = p_1^2 p_2$, $n = p_1 p_2$ where p_1, p_2 are distinct primes.*

Proof. If we assume that $n > 1$ we have $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime factorisation of $n > 1$. It is well-known that

$$\tau^{**}(n) = \left\{ \prod_{\alpha_i = 2k} \alpha_i \right\} \left\{ \prod_{\alpha_i = 2k+1} (\alpha_i + 1) \right\}, \tag{1}$$

(see [11]).

We have by Lemma 2.1 that n multiplicatively bi-unitary perfect iff

$$n^{\frac{\tau^{**}(n)}{2}} = n^2$$

or

$$\tau^{**}(n) = 4,$$

$$\left\{ \prod_{\alpha_i = 2k} \alpha_i \right\} \left\{ \prod_{\alpha_i = 2k+1} (\alpha_i + 1) \right\} = 4$$

which equivalent to the forms $n = p_1^4$, $n = p_1^3$, $n = p_1^2 p_2^2$, $n = p_1^2 p_2$, $n = p_1 p_2$. where p_1, p_2 are distinct primes.

□

Theorem 2.3 *All m bi-unitary super perfect numbers n, $(n > 1)$ have one of the following forms: $n = p$, $n = p^2$ where p prime number.*

Proof. Let $n > 1$ a natural number. We have

$$T^{**}(T^{**}(n)) = (T^{**}(n))^{\frac{\tau^{**}(T^{**}(n))}{2}} = n^{\frac{\tau^{**}(n)}{2}\frac{\tau^{**}(T^{**}(n))}{2}}. \qquad (2)$$

Because $n > 1$

$$\tau^{**}(T^{**}(n)) = \tau^{**}\left(n^{\frac{\tau^{**}(n)}{2}}\right).$$

- If $\tau^{**}(n) = 2(2k+1)$,

$$\begin{aligned} \tau^{**}(T^{**}(n)) &= \tau^{**}\left(n^{2k+1}\right) = \left\{\prod_{\alpha_i=2k}(2k+1)\alpha_i\right\}\left\{\prod_{\alpha_i=2k+1}(2k+1)(\alpha_i+1)\right\} \\ &= (2k+1)^{\omega(n)}\tau^{**}(n) = 2(2k+1)^{\omega(n)+1} \qquad (3) \end{aligned}$$

- If $\tau^{**}(n) = 4k$,

$$\begin{aligned} \tau^{**}(T^{**}(n)) &= \tau^{**}\left(n^{2k}\right) = \left\{\prod_{\alpha_i} 2k\alpha_i\right\} \\ &= k^{\omega(n)}\tau^{**}(n^2) \qquad (4) \end{aligned}$$

From (2) we have that n bi-unitary m superperfect number iff

$$\frac{\tau^{**}(n)}{2}\cdot\frac{\tau^{**}(T^{**}(n))}{2} = 2.$$

If $\tau^{**}(n) = 2(2k+1)$,

$$(2k+1)^{\omega(n)+2} = 1$$

which imlies that $k = 0$, $\tau^{**}(n) = 2$; $n = p$ or $n = p^2$.

If $\tau^{**}(n) = 4k$, we have

$$k^{\omega(n)+1}\tau^{**}(n^2) = 2.$$

Because $n > 1$, $k = 0$ and $n = p$.

□

We now investigate the problem of the existence of $k - l$ bi-unitary m perfect numbers.

First we prove that:

Lemma 2.4 *If* n *a natural number such that* $\omega(n) \geq l + 1$ *we have:*

$$T^{**k}(n) \geq n^{2^{lk}}.$$

Proof Because $\omega(n) \geq l + 1$ we have

$$\tau^{**}(n) \geq 2^{l+1}$$

which together to Lemma 2.1 implies that

$$T^{**}(n) \geq n^{2^{l}}. \tag{5}$$

By (5) and iteration procedure we have

$$T^{**k}(n) \geq n^{2^{lk}}.$$

□

Remark If $\omega(n) \geq 2$ we have

$$T^{**k}(n) \geq n^{2^{k}}.$$

which means that if $l \leq 2^k$ no (k,l) m bi-unitary perfect composed number.

In the next we consider the $(1, k)$ m bi-unitary perfect numbers:

$$T^{**}(n) = n^k,$$

where $k \geq 2$ is given.

From the lemma 2.1 and (1) we have the following theorem.

Theorem 2.5 *Let* $k \geq 2$ *be a natural number. Then the* $(1, k)$ *m bi-unitary perfect numbers have the form*

$$n = \prod_{k_i = 2k} p_i^{k_i} \prod_{k_i = 2k+1} p_i^{k_i - 1}$$

where $k = \prod_{i=1}^{l} k_i$ *and* p_i $(i \in \{1, 2, \ldots, l\})$ *distinct prime numbers.*

Concerning even perfect numbers we have a following result.

Theorem 2.6 *For every even perfect number exist* k *such that* n m *bi-unitary* $(1, k)$ *perfect number.*

Proof. By Lemma 1.1 n even perfect iff $n = 2^{p-1}q$, where $q = 2^p - 1$ is prime which implies that $\tau^{**}(n) = 2(p-1)$ and

$$T^{**}(n) = n^{\left(\frac{\tau^{**}(n)}{2}\right)} = n^{p-1} = n^k$$

with $k = p - 1$.

□

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