q-BERNOULLI NUMBERS AND POLYNOMIALS VIA AN INVARIANT p-ADIC q-INTEGRAL ON $\mathbb{Z}_p$

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Abstract. We define the q-Bernoulli numbers by using an p-adic q-integral due to T. Kim (see [2]) and investigate the properties of these numbers. In the final section, we will give the formula for sums of products of these numbers.

§1. Introduction

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$.

Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. If $q \in \mathbb{C}_p$, we normally assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to 1}[x : q] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case.

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Recently, I. C. Huang [3] and K. Dilcher [2] obtained formulas for sums of products of the form $\sum \binom{2n}{2j_1, \cdots, 2j_N} B_{2j_1} \cdots B_{2j_N}$ and J. Satoh [6] gave a formulas for sums of products in the case of two Carlitz's q-Bernoulli numbers.

In this paper, we define the q-Bernoulli numbers of higher order by using multiple p-adic q-ntegral due to T. Kim [4] and will give the formulas for sums of products of any number of the our q-Bernoulli numbers which is induced the formulae of Dilcher([2]), I. C. Huang ([3]) at $q = 1$.

§2. On q-Bernoulli numbers

Let d be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N : q]}$$

can be extended to distribution on X [4].

This distribution yields an integral for each non-negative integer m [4]:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_q(a) = \int_X f(x)\, d\mu_q(a) = \lim_{n\to\infty} \frac{1}{[dp^N]} \sum_{N=0}^{dp^N-1} f(x) q^x, \tag{1}$$

where $f(x)$ is the uniformly differentiable function.

In this section, we can consider a uniformly differentiable function $f(x) = x^n$, $(n \geq 0)$, in the p-adic q-integral given by (1). Now, we define q-Bernoulli numbers and polynomials as follows:

$$\beta_n = \int_{\mathbb{Z}_p} t^n d\mu_q(t) \text{ and } \beta_n(x, q) = \int_{\mathbb{Z}_p} (x + t)^n d\mu_q(t).$$

in the variable x in $\mathbb{C}_p$ with $|x|_p \leq 1$.

For $k \in \mathbb{N} =$ { the set of natural numbers }, it was well known that the Bernoulli numbers with order k were defined by

$$(\frac{t}{e^t-1})^k = \sum_{n=0}^{\infty} \frac{B_n^{(k)}}{n!} t^n, \tag{2}$$

where $B_n^{(k)}$ are called n-th Bernoulli numbers with order k, (cf. [2], [3], [5]).

Now, we define the q-Bernoulli numbers and polynomials of higher order, $\beta_m^{(k)}$ and $\beta_m^{(k)}(x,q) \in \mathbb{C}_p$, by using Kim's integral as follows:

$$\beta_m^{(k)} = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k \text{ times}} (t_1+t_2+\cdots+t_k)^m d\mu_q(t_1)d\mu_q(t_2)\cdots d\mu_q(t_k),$$

$$\beta_m^{(k)}(x,q) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k \text{ times}} (x+t_1+t_2+\cdots+t_k)^m d\mu_q(t_1)d\mu_q(t_2)\cdots d\mu_q(t_k).$$

Let $F_q(t)$ be the generating function of β_n in the above which is presented by

$$F_q(t) = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} t^n.$$

It is not difficult to see that

$$F_q(t) = (\frac{q-1}{\log q})\frac{\log q + t}{qe^t - 1} \text{ and } F_q(t)e^{xt} = \sum_{n=0}^{\infty} \frac{\beta(x,q)}{n!} t^n. \tag{3}$$

In [4], an invariant p-adic integral on $\mathbb{Z}_p$ can be found by

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x).$$

Note that

$$I_1(f_1) = I_1(f) + f'(0), \tag{4}$$

where $f_1(x) = f(x+1)$, (cf. [4]).

If we take $f(x) = q^x e^{xt}$, we can prove (3) from (4) and [4].

For any positive integer d and $k \geq 0$, it is easy to see that

$$\beta_k(x,q) = \frac{d^k}{[d]} \sum_{i=0}^{d-1} q^i \beta_k(\frac{x+i}{d}, q^d), \tag{5}$$

(q-Bernoulli distribution).

By using the definition of q-Bernoulli numbers of higher order, we see:

$$\begin{aligned}\beta_k^{(l)} &= \lim_{t\to\infty} \frac{t}{[m]} \frac{l}{[p^t : q^m]} \sum_{i_1,\cdots,i_l=0}^{m-1} \sum_{n_1,\cdots,n_l=0}^{p^t-1} q^{\sum_{j=1}^{l} i_j + m\sum_{j=1}^{l} n_j} (x + \sum_{j=1}^{l} i_j + m \sum_{j=1}^{l} n_j)^k \\ &= \frac{m^k}{[m]^l} \sum_{i_1,\cdots,i_l=0}^{m-1} q^{i_1+\cdots+i_l} \beta_k^{(l)}(\frac{i_1+\cdots+i_l}{m}, q^m).\end{aligned}$$

Here, we use the following notation:

$$\sum_{k_1=0}^{m} \sum_{k_2=0}^{m} \cdots \sum_{k_n=0}^{m} = \sum_{k_1,\cdots,k_n=0}^{m}.$$

Note that $\lim_{q\to 1} \beta_m^{(k)}(q) = B_m^{(k)}$ where $B_m^{(k)}$ is the mth Bernoulli number with order k, (cf. [2], [3], [6]).

Therefore we obtain the following theorem:

Theorem 1. *For any positive integers m, k, we have*

$$\beta_k^{(l)}(x,q) = \frac{m^k}{[m]^l} \sum_{i_1,\cdots,i_l=0}^{m-1} q^{i_1+\cdots+i_l} \beta_k^{(l)}(\frac{i_1+\cdots+i_l}{m}, q^m).$$

In particular,

$$\beta_k^{(l)}(mx,q) = \frac{m^k}{[m]^l} \sum_{i_1,\cdots,i_l=0}^{m-1} q^{i_1+\cdots+i_l} \beta_k^{(l)}(x + \frac{i_1+\cdots+i_l}{m}, q^m).$$

For $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{C}_p$ and positive integers n, m, we have

$$(6)\quad (\alpha_1 \cdots + \alpha_m + t_1 + \cdots + t_m)^n = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1, \cdots, i_m} (t_1+\alpha_1)^{i_1} \cdots (t_m+\alpha_m)^{i_m}.$$

By (6) and the definition of q-Bernoulli polynomials, we have

$$\beta_n^{(m)}(\alpha_1 + \cdots + \alpha_m, q) = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1, \cdots, i_m} \beta_{i_1}(\alpha_1, q) \cdots \beta_{i_m}(\alpha_m, q),$$

where $\binom{n}{i_1, \cdots, i_m}$ are multinomial coefficients.

Therefore we obtain the following:

Theorem 2. *For $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{C}_p$ and positive integers n, m, we have*

$$\beta_n^{(m)}(\alpha_1 + \alpha_2 + \cdots + \alpha_m, q) = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1, \cdots, i_m} \beta_{i_1}(\alpha_1, q) \cdots \beta_{i_m}(\alpha_m, q),$$

where $\binom{n}{i_1, \cdots, i_m}$ are multinomial coefficients.

Corollary 3. *For any positive integers m, n, we have*

$$\beta_n^{(m)}(q) = \sum_{\substack{i_1, \cdots, i_m \geq 0 \\ i_1+\cdots+i_m=n}} \binom{n}{i_1, \cdots, i_m} \beta_{i_1}(q)\beta_{i_2}(q) \cdots \beta_{i_{m-1}}(q)\beta_{i_m}(q)$$

Remark. If $q \to 1$, then we obtain the following [6]:

(1) $\quad B_m^{(k)}(k - x) = (-1)^m B_m^{(k)}(x).$

(2) $\quad B_m^{(k)}(k) = (-1)^m B_m^{(k)}.$

Remark. If $q \to 1$, we obtain the following formula (cf. [2], [3]):

$$B_n^{(m)}(\alpha_1 + \alpha_2 + \cdots + \alpha_m) = \sum_{\substack{i_1, \cdots, i_m \geq 0 \\ i_1+\cdots+i_m=n}} \binom{n}{i_1, \cdots, i_m} B_{i_1}(\alpha_1)B_{i_2}(\alpha_2) \cdots B_{i_m}(\alpha_m).$$

References

1. L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math J. **15** (1948), 987-1000.

2. K. Dilcher, *Sums of products of Bernoulli numbers*, J. Number Theory **60(1)** (1996), 23–41.

3. I. C. Huang, *Bernoulli numbers and polynomials via residue*, J. Number Theory **76** (1999), 178–193.

4. T. Kim, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory **76** (1999), 320–329.

5. T. Kim and H.S. Kim, *Remark on p-adic q-Bernoulli numbers*, Advan. Stud. Contemp. Math. **1** (1999), 127-136.

6. K. Kudo, *A congruence of generalized Bernoulli numbers for the character of the first kind*, Advan. Stud. Contemp. Math. **2** (2000), 1–8.

7. J. Satoh, *Sums of products of two q-Bernoulli numbers*, J. Number Theory **74** (1999), 173–180.