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A NOTE ON THE ANALOGS OF p-ADIC L-FUNCTIONS AND SUMS OF POWERS

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ABSTRACT. The purpose of this paper is to give an explicit p-adic expansion of ${\sum^*}_{j=1}^{np} \frac{q^j}{j^r}$ such that the coefficients of the expansion are the values of an analogue of p-adic L-function associated with Euler numbers.

1. INTRODUCTION

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously

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considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. We use the notation:

$$[x] = [x : q] = \frac{1-q^x}{1-q}. \tag{1}$$

Note that when p is prime $[p]$ is an irreducible polynomial in $Q(q)$. Furthermore, this means that $Q(q)/[p]$ is a field and consequently rational functions $r(q)/s(q)$ are well defined modulo $[p]$ if $(r(q), s(q)) = 1$. Recently Andrews (see [1]) presented q-analogs of several classical binomial coefficient congruences due to Babbage, Wolstenholme and Glaisher. In [5], the first author has given an explicit formulas to generalize the theorem of Andrews. In [11], L.C. Washington has given an explicit p-adic expansion of $\sum_{j=1,(j,p)=1}^{np} \frac{1}{j^r}$ as power series in n .

In this paper, we give an explicit p-adic expansion of $\sum_{j=1,(j,p)=1}^{np} \frac{q^j}{j^r}$ such that the coefficients of the expansion are the values of an analogue of p-adic L-function associated with Euler numbers by more or less the same method in [5]. In Section 2, we give the new identities for the analogs of Bernoulli numbers, which were studied by the first author in [9]. These identities will be used to give an explicit p-adic expansion of $\sum_{j=1,(j,p)=1}^{np} \frac{q^j}{j^r}$ as power series in n.

2. On identities of the analogue of Bernoulli numbers

The Bernoulli numbers B_m are defined by means of the exponential generating function

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m, \quad |t| < 1. \tag{2}$$

It is easy to see that

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6, \text{ and } B_{2k+1} = 0 \text{ for } k > 0.$$

For any positive integer N, $\mu_0(a + p^N\mathbb{Z}_p) = \frac{1}{p^N}$ can be extended to a distribution on $\mathbb{Z}_p$, cf. [2]. Let $T_p = \lim_{n\to\infty} C_{p^n}$, where C_{p^n} are denoted by cyclic group of order p^n.

For $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, $q(\neq 1) \in T_p$, we define the the analogue of Bernoulli numbers, $\beta_m = \beta_m(q)$ as

$$\beta_m = \int_{\mathbb{Z}_p} q^x x^m d\mu_0(x). \tag{3}$$

Thus we have, cf. [9],

$$q(\beta+1)^k - \beta_k = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k>1, \end{cases} \tag{4}$$

with the usual convention about replacing β^i by β_i.

So, the analogue of Bernoulli numbers β_m are defined by means of the generating function $G_q(t)$ as follows:

$$G_q(t) = \frac{t}{qe^t - 1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} t^n, \tag{5}$$

where $t \in \mathbb{Z}_p$, $q(\neq 1) \in T_p$.

The analogue of Bernoulli polynomials in the variable x in $\mathbb{C}_p$ with $|x|_p \leq 1$ were defined by

$$\beta_n(x,q) = \int_{\mathbb{Z}_p} q^t (x+t)^n d\mu_0(t), \quad \text{for } q(\neq 1) \in T_p, \text{ (see [2], [3], [9])}. \tag{6}$$

It is easy to see in [9] that

$$\beta_n(x,q) = (\beta+x)^n = \sum_{j=0}^{n} \binom{n}{j} \beta_j x^{n-j}, \quad \text{for } q(\neq 1) \in T_p. \tag{7}$$

Let $G_q(x,t)$ be the generating function of the analogue of Bernoulli polynomials in the variable x. By (7), (5), we see that

$$G_q(x,t) = \frac{t}{qe^t - 1} e^{xt}, \quad \text{for } t \in \mathbb{Z}_p,\ q(\neq 1) \in T_p. \tag{8}$$

By (8), it is shown that

$$d^{k-1} \sum_{i=0}^{d-1} \beta_k\left(\frac{x+i}{d}, q^d\right) q^i = \beta_k(x,q), \tag{9}$$

where d, k are positive integers.

From now, we assume $q \in \mathbb{C}$ with $|q| < 1$, $q \neq 1$ and we try to prove the q-analogue of (1) by using (9).

If we put $x = 0$ in (9), then we have

$$n\beta_m = \sum_{k=0}^{m} \binom{m}{k} \beta_k(q^n) n^k \sum_{j=0}^{n-1} q^j j^{m-k}. \tag{10}$$

By (10), we see

$$n\beta_m - \beta_m(q^n) n^m [n] = \sum_{k=0}^{m-1} \binom{m}{k} n^k \beta_k(q^n) \sum_{j=1}^{n-1} q^j j^{m-k}. \tag{11}$$

Define the operation $*$ on $f(q)$ as follows:

$$n(1 - n^m) * f(q) = nf(q) - n^m [n] f(q^n). \tag{12}$$

Thus (11) can be written using $*$ as:

$$n(1 - n^m) * \beta_m(q) = \sum_{k=0}^{m-1} \binom{m}{k} n^k \beta_k(q^n) \sum_{j=1}^{n-1} q^j j^{m-k}. \tag{13}$$

It was well known that, for positive integers s and n,

$$\sum_{l=0}^{n-1} l^{s-1} = \frac{1}{s} \sum_{j=0}^{s-1} \binom{s}{j} B_j n^{s-j}. \tag{14}$$

Now, we would like to give the analogue of (14) which is used later. It is easy to see that

$$\sum_{l=0}^{n-1} e^{lt} q^l = \frac{1}{t} \frac{(q^n e^{nt} - 1)t}{qe^t - 1}, \quad \text{where } q \in \mathbb{C} \text{ with } |q| < 1 . \tag{15}$$

By (5), (8), (15), we have the following formula:

$$q^n \beta_m(n, q) - \beta_m = m \sum_{l=0}^{n-1} q^l l^{m-1}.$$

Hence, we can give the q-analogue of (14) as follows:

$$\sum_{l=0}^{n-1} q^l l^{m-1} = \frac{q^n}{m}\sum_{l=0}^{m-1}\binom{m}{l}\beta_l n^{m-l} + \frac{1}{m}(q^n - 1)\beta_m. \tag{16}$$

It can be defined by more or less the same method in [7], [10] that

$$\zeta_q(s) = \sum_{n=1}^{\infty}\frac{q^n}{n^s}, \quad \zeta_q(s,a) = \sum_{n=0}^{\infty}\frac{q^n}{(n+a)^s}, \tag{17}$$

where a is a real number with $0 < a \leq 1$, and $q \in \mathbb{C}$ with $|q| < 1$, $s \in \mathbb{C}$.

Note that

$$\zeta_q(1-k) = -\frac{\beta_k}{k}, \quad \zeta_q(1-k,a) = -\frac{\beta_k(a,q)}{k}, \tag{18}$$

where k is any positive integer.

Let

$$\begin{aligned} J_q(s,a,F) &= \sum_{\substack{m\equiv a \pmod F \\ m>0}} \frac{q^m}{m^s} = \sum_{n=0}^{\infty}\frac{q^{a+nF}}{(a+nF)^s} \\ &= \frac{q^a}{F^s}\sum_{n=0}^{\infty}\frac{q^{nF}}{(\frac{a}{F}+n)^s} = \frac{q^a}{F^s}\zeta_{q^F}(s,\frac{a}{F}), \end{aligned} \tag{19}$$

where a and F are positive integers with $0 < a < F$.

Then

$$J_q(1-n,a,F) = -\frac{F^{n-1}q^a\beta_n(\frac{a}{F},q^F)}{n}, \qquad n \geq 1, \tag{20}$$

and J_q has a simple pole at $s = 1$.

Let χ be the Dirichlet character with conductor F. Then we define the analogue of the Dirichlet L-series as follows: For $s \in \mathbb{C}$,

$$l_q(s,\chi) = \sum_{a=1}^{F}\chi(a)J_q(s,a,F).$$

Now, we define the analogue of generalized Bernoulli numbers with χ as follows:

$$\sum_{a=1}^{F-1} \frac{tq^a\chi(a)e^{at}}{q^F e^{Ft}-1} = \sum_{n=0}^{\infty} \frac{\beta_{n,\chi}}{n!} t^n,$$

where $q(\neq 1) \in \mathbb{C}$ with $|q|<1$.

Hence, we have

$$l_q(1-k,\chi) = -\frac{\beta_{k,\chi}}{k}, \quad \text{for } k \geq 1.$$

Remark. For $q(\neq 1) \in \mathbb{C}$ with $|q|<1$, it is easy to see that

$$\frac{\beta_m}{m} = \frac{q^{-1}}{1-q^{-1}} H_m(q^{-1}), \quad \text{cf. [9]},$$

where $H_m(q^{-1})$ are Euler numbers.

3. An Analogue of p-adic L-function

In this section, we assume $q \in T_p$. Let p be an odd prime and let $l_{p,q}(s,\chi)$ be the q-analogue of the p-adic L-function attached to a character χ which is defined late. We define $<x>=<x:q>=\frac{x}{\omega(x)}$, where $\omega(x)$ is the Teichmüller character. When F is a multiple of p and $(a,p)=1$, we define a p-adic analogue of (19) as

$$J_{p,q}(s,a,F) = \frac{1}{s-1}\frac{q^a}{F} <a>^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} \left(\frac{F}{a}\right)^j \beta_j(q^F), \tag{21}$$

for $s \in \mathbb{Z}_p$.

It is easy to see in (7), (20), (21) that

$$\begin{aligned} J_{p,q}(1-n,a,F) &= -\frac{1}{n}\frac{q^a}{F} <a>^n \sum_{j=0}^{n} \binom{n}{j} \beta_j(q^F) \left(\frac{F}{a}\right)^j \\ &= -\frac{1}{n} F^{n-1} q^a \omega^{-n}(a) \sum_{j=0}^{n} \binom{n}{j} \beta_j(q^F) \left(\frac{a}{F}\right)^{n-j} \\ &= -\frac{1}{n} F^{n-1} \omega^{-n}(a) q^a \beta_n\left(\frac{a}{F}, q^F\right) = \omega^{-n}(a) J_q(1-n,a,F), \end{aligned} \tag{22}$$

for all positive integers n and it has a simple pole at $s = 1$.

It is easy to see from [3], (21), (20) that

$$l_{p,q}(s,\chi) = \sum_{\substack{a=1 \\ p\nmid a}}^{F} \chi(a) J_{p,q}(s,a,F). \tag{23}$$

For $f \in \mathbb{N}$, let χ be the Dirichlet character with conductor f. The analogue of generalized Bernoulli numbers with character χ, which were defined in Section 2, is associated with an invariant p-adic integral as follows:

$$\beta_{k,\chi}(q) = \int_{\mathbb{Z}_p} \chi(x) q^x x^k d\mu_0(x), \quad \text{cf. [9]}, \tag{24}$$

where k is a positive integer, $q(\neq 1) \in T_p$.

We see in (24) that

$$\beta_{k,\chi}(q) = f^{k-1} \sum_{a=1}^{f} \chi(a) q^a \beta_k \left(\frac{a}{f}, q^f \right). \tag{25}$$

By (22), (23), (25), if $n \geq 1$ then we have

$$\begin{aligned} l_{p,q}(1-n,\chi) &= \sum_{\substack{a=1 \\ p\nmid a}}^{F} \chi(a) J_{p,q}(1-n,a,F) \\ &= -\frac{1}{n}(\beta_{n,\chi\omega^{-n}}(q) - p^{n-1}\chi\omega^{-n}(p)\beta_{n,\chi\omega^{-n}}(q^p)). \end{aligned} \tag{26}$$

In fact, we have the formula

$$l_{p,q}(s,\chi) = \frac{1}{F}\frac{1}{s-1} \sum_{\substack{a=1 \\ p\nmid a}}^{F} \chi(a) q^a < a >^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} \beta_j(q^F) \left(\frac{F}{a}\right)^j, \tag{27}$$

for $s \in \mathbb{Z}_p$.

This is a p-adic analytic function (except possibly at $s = 1$) and has the following properties of (28)-(31) for $\chi = \omega^t$ in [7], [9] by more or less the same method:

$$l_{p,q}(1-k,\omega^t) = -\frac{1}{k}(\beta_k - p^{k-1}\beta_k(q^p)), \tag{28}$$

where $1 \leq k \equiv t \pmod{p-1}$,

(29) $$l_{p,q}(s, \omega^t) \in \mathbb{Z}_p \qquad \text{for all } s \in \mathbb{Z}_p \text{ when } t \not\equiv 0 \pmod{p-1}.$$

If $t \not\equiv 0 \pmod{p-1}$, then

(30) $$l_{p,q}(s_1, \omega^t) \equiv l_{p,q}(s_2, \omega^t) \pmod{p} \qquad \text{for all } s_1, s_2 \in \mathbb{Z}_p.$$

$l_{p,q}(s, 1)$ has a simple pole at $s = 1$ with residue 0,

(31) $$l_{p,q}(k, \omega^t) \equiv l_{p,q}(k + p, \omega^t) \pmod{p}.$$

It was known in [4], [9] that

(32) $$\frac{1}{r+k-1}\binom{-r}{k}\binom{1-r-k}{j} = \frac{-1}{j+k}\binom{-r}{k+j-1}\binom{k+j}{j},$$

for all positive integers r, j, k with $j, k \geq 0$, $j + k > 0$, and $r \neq 1 - k$.

Thus we can obtain the following: For $r \geq 1$,

(33) $$\begin{aligned}&\sum_{k=1}^{\infty}\binom{-r}{k}\omega^{1-r-k}(a)q^n J_{p,q}(r+k, a, F)(Fn)^k \\ &= -\sum_{l=0}^{n-1}\frac{q^{Fl+a}}{(Fl+a)^r} - (q^{Fn}-1)\sum_{s=1}^{\infty} q^a a^{-r}\left(\frac{F}{a}\right)^{s-1}\binom{-r}{s-1}\frac{\beta_s(q^F)}{s}.\end{aligned}$$

For $F = p$, $r \in \mathbb{N}$, we see that

(34) $$\sum_{a=1}^{p-1}\sum_{l=0}^{n-1}\frac{q^{a+pl}}{(a+pl)^r} = \sum_{j=1,(j,p)=1}^{np}\frac{q^j}{j^r}.$$

We set

(35) $$B^{(r)}(a, F) = \sum_{s=1}^{\infty} q^a a^{-r}\left(\frac{F}{a}\right)^{s-1}\binom{-r}{s-1}\frac{\beta_s(q^F)}{s}.$$

By (33), (34), (35), we have

(36)
$$\sum_{j=1}^{np}{}^{*}\frac{q^j}{j^r} = -\sum_{k=1}^{\infty}\binom{-r}{k}q^n(pn)^k l_{p,q}(r+k, \omega^{1-k-r}) - (q^{pn}-1)\sum_{a=1}^{p-1}\omega^{1-k-r}(a)B^{(r)}(a, F).$$

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References

1. G.E. Andrews, *q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher*, Discrete Math. **204** (1999), 15-25.

2. T. Kim, *On a q-analogue of p-adic* log *gamma functions and related integrals*, J. Number Theory **78** (1999), 320–329.

3. ______, *A note on p-adic Carlitz's q-Bernoulli numbers*, Bulletin Austral. Math.. **62** (2000), 227-234.

4. ______, *Sums products of q-Bernoulli numbers*, Arch. Math. **76 no. 3** (2001), 190-195.

5. ______, *On p-adic q-L-functions and sums of powers*, Discrete Math. (2001).

6. ______, *A note on p-adic q-Dedekind sums*, Comptes Rendus de l' Academie Bulgare des Sciences **54 no. 10** (2001), 37-42.

7. ______, *Remark on p-adic q-Bernoulli numbers*, Adv. Stud. Contemp. Math.(Kudeok Publ.) **1** (1999), 127-136.

8. ______, *Some q-Bernoulli numbers of higher order associated with the p-adic q-integrals*, Indian J. Pure and Appl. Math. **32 no. 10** (2001), 1565-1570.

9. ______, *An analogue of Bernoulli numbers and their congruences*, Rep. Fac. Sci. Engrg. Saga Univ. Math. **22** (1994), 7-13.

10. L.C. Jang, T. Kim, D-H. Lee, D-W. Park, *An application of polylogarithms in the analogs of Genocchi numbers*, Notes on Number Theory and Discrete Mathematics **7 no. 3** (2001), 65-69.

11. L. C. Washington, *p-adic L-functions and sums of powers*, J. Number Theory **69** (1998), 50-61.