NNTDM 7 (2001) 2, 61-64

ON 28-th SMARANDACHE'S PROBLEM

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The 28-th problem from [1] (see also 94-th problem from [2]) is the following:

Smarandache odd sieve:

$$7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, 67, 73, 75, 83, 85, 91, 93, 97, \ldots$$

(All odd numbers that are not equal to the difference of two primes).
A sieve is used to get this sequence:
- substract 2 from all prime numbers and obtain a temporary sequence;
- choose all odd numbers that do not belong to the temporary one.

We find an explicit form of the n-th term of the above sequence, that will be denoted by $C = \{C_n\}_{n=1}^{\infty}$ below. Let $\pi_C(n)$ be the number of the members of the terms of C which are not greater than n. In particular, $\pi_C(0) = 0$.

Firstly, we shall note that the above definition of C can be interpreted to the following equivalent form as follows, having in mind that every odd number is a difference of two prime numbers if and only if it is a difference of a prime number and 2:

Smarandache's odd sieve contains exaclty these odd numbers which cannot be represented as a difference of a prime number and 2.

We can rewrite the last definition to the following equivalent form, too:

Smarandache's odd sieve contains exaclty these odd numbers which are represented as a difference of a composite odd number and 2.

We shall find an explicit form of the n-th term of the above sequence, using the third definition of it. Initially, we shall prove the following two lemmas.

LEMMA 1: For every natural number $n \geq 1$, C_{n+1} is exactly one of the numbers: $u \equiv C_n + 2$, $v \equiv C_n + 4$ or $w \equiv C_n + 6$.

Proof: Let us assume that none of the numbers u, v, w coincides with C_{n+1}. Having in mind the third form of the above definition, number u is composite and by assumption u is not a member of sequence C. Therefore v, according to the third form of the definition is a prime number and by assumption it is not a member of sequence C. Finally, w, according to the third form of the definition is a prime number and by assumption it is not a member of sequence C. Therefore, according to the third form of the definition number $w + 2$ is prime.

Hence, from our assumptions we obtained that all of the numbers v, w and $w + 2$ are prime, which is impossible, because these numbers are consequtive odd numbers and having in mind that $v = C_n + 4$ and $C_1 = 7$, the smallest of them satisfies the inequality

$v \geq 11$.
COROLLARY For every natural number $n \geq 1$:

$$C_{n+1} \leq C_n + 6. \tag{1}$$

LEMMA 2: For every natural number $n \geq 1$:

$$C_n \leq 6n + 1. \tag{2}$$

Proof: We use induction. For $n = 1$ obviously we have equality. Let us assume that (2) holds for some n. We shall prove that

$$C_{n+1} \leq 6(n+1) + 1. \tag{3}$$

By (1) and the induction assumption it follows that

$$C_{n+1} \leq C_n + 6 \leq (6n+1) + 6 = 6(n+1) + 1,$$

which proves (3).

Now, we return to the Smarandache's problem.

Let $\pi_C(N)$ be the number of the members of the sequence $\{C_n\}_{n=1}^{\infty}$ not bigger than N. In particular, $\pi_C(0) = 0$.

In [3] the following three universal formulae are introduced in an expicit form, using numbers $\pi_C(k)$ $(k = 0, 1, 2, ...)$, which can be used for the present numbers C_n:

$$C_n = \sum_{k=0}^{\infty} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}], \tag{4}$$

$$C_n = -2. \sum_{k=0}^{\infty} \zeta(-2.[\frac{\pi_C(k)}{n}]), \tag{5}$$

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}, \tag{6}$$

where $[x]$ is the largest integer not greater than the real nonnegative number x; ζ is the Riemann's function zeta; Γ is the Euler's function gamma.

For the present case, having in mind (2), we can substitute symbol ∞ with $6n + 1$ in sum $\sum\limits_{k=0}^{\infty}$ for C_n and we obtain the following sums:

$$C_n = \sum_{k=0}^{6n+1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}], \tag{7}$$

$$C_n = -2. \sum_{k=0}^{6n+1} \zeta(-2.[\frac{\pi_C(k)}{n}]), \tag{8}$$

$$C_n = \sum_{k=0}^{6n+1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \quad (9)$$

We must show why $\pi_C(n)$ $(n = 1, 2, 3, ...)$ is represented in an explicit form. It can be directly seen that the number of the odd numbers not bigger that n is exactly equal to

$$\alpha(n) = n - [\frac{n}{2}], \quad (10)$$

because the number of the even numbers not bigger that n is exactly equal to $[\frac{n}{2}]$.

Let us denote by $\beta(n)$ the number of all odd numbers not bigger that n, which can be represented as a difference of two primes. According the second form of the definition, giving above, $\beta(n)$ coincides with the number of all odd numbers m such that $m \leq n$ and m has the form $m = p - 2$, where p is an odd prime number. Therefore, we much study all odd prime numbers, because of the inequality $m \leq n$. The number of these prime numbers is exactly $\pi(n+2) - 1$. Therefore,

$$\beta(n) = \pi(n+2) - 1. \quad (11)$$

Omitting from the number of all odd numbers not bigger that n the number of those which are a difference of two primes, we find exactly the number of these odd numbers not bigger that n which are not a difference of two prime numbers, i.e., $\pi_C(n)$. Therefore, the equality

$$\pi_C(n) = \alpha(n) - \beta(n)$$

holds and from (10) and (11) we obtain:

$$\pi_C(n) = (n - [\frac{n}{2}]) - (\pi(n+2) - 1) = n + 1 - [\frac{n}{2}]) - \pi(n+2),$$

where $\pi(m)$ is the number of primes p such that $p \leq m$. But $\pi(n+2)$ can be represented in an explicit form, e.g., by Mináč's formula (see [4]):

$$\pi(n+2) = \sum_{k=2}^{n+2} [\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]],$$

and therefore, we obtain that the explicit form of $\pi_C(N)$ is

$$\pi_C(N) = N + 1 - [\frac{N}{2}]) - \sum_{k=2}^{N+2} [\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]], \quad (12)$$

where $N \geq 1$ is a fixed natural number.

It is possible to put $[\frac{N+3}{2}]$ instead of $N + 1 - [\frac{N}{2}]$ into (12).

Now, using each of the formulae (7) - (9), we obtain C_n in an explicit form, using (12).

It can be checked directly that

$$C_1 = 7,\ C_2 = 13,\ C_3 = 19,\ C_4 = 23,\ C_5 = 25,\ C_6 = 31,\ C_7 = 33, ...$$

and

$$\pi_C(0) = \pi_C(1) = \pi_C(2) = \pi_C(3) = \pi_C(4) = \pi_C(5) = \pi_C(6) = 0.$$

Therefore from (7)-(9) we have the following explicit formulae for C_n

$$C_n = 7 + \sum_{k=7}^{6n+1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}],$$

$$C_n = 7 - 2. \sum_{k=7}^{6n+1} \zeta(-2.[\frac{\pi_C(k)}{n}]),$$

$$C_n = 7 + \sum_{k=7}^{6n+1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])},$$

where $\pi_C(k)$ is given by (12).

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