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Expansion of Integer Powers from Fibonacci's Odd Number Triangle

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Abstract

Cubes and squares are expanded in various ways stimulated by Fibonacci's odd number triangle which is in turn extended to even powers. The class structure of the cubes within the modular ring $\mathbb{Z}_4$ is developed. This provides constraints for the various functions which help in solving polynomial and diophantine equations, some simple examples of which are given.

1. Introduction

One of the most remarkable results of that remarkable medieval mathematician, Leonardo Fibonacci of Pisa, was his triangle of odd numbers which is displayed in Table 1 (Hollingdale, 1989: 103).

1^3	1										
2^3	3	5									
3^3	7	9	11								
4^3	13	15	17	19							
5^3	21	23	25	27	29						
6^3	31	33	35	37	39	41					
7^3	43	45	47	49	51	53	55				
8^3	57	59	61	63	65	67	69	71			
9^3	73	75	77	79	81	83	85	87	89		
10^3	91	93	95	97	99	101	103	105	107	109	
11^3	111	113	115	117	119	121	123	125	127	129	131

Table 1: Fibonacci's Odd Number Triangle

If we represent the entries in Table 1 by $u_{i,j}$, $i,j \geq 1$, then

$$u_{i,j} = i^2 - i + 2j - 1, \tag{1.1}$$

$$i^3 = \sum_{j=1}^{i} u_{i,j}, \tag{1.2}$$

$$u_{i,j} - u_{i-1,j-1} = 2i, \tag{1.3}$$

$$u_{i,j} - u_{i-1,j} = 2i - 2, \tag{1.4}$$

$$u_{i,1} = i^2 - i + 1, \tag{1.5}$$

$$u_{i,i} = i^2 + i - 1, \tag{1.6}$$

where $\{u_{i,1}\}$ is the sequence of central polygonal numbers (Hogben, 1950: 22), and $\{u_{i,i}\}$ contains the sequence of primes generated by (1.5) (Lehmer, 1941: 46). The results are also related to aspects of the studies by Utz (1977), Wieckowski (1980) and Ando (1982).

By adding the rows of Table 1, we observe that we get the known result

$$\sum_{j=1}^{N} j^3 = \left(\sum_{j=1}^{N} j\right)^2, \tag{1.7}$$

the successive terms within the right hand bracket being the triangular numbers $\frac{1}{2}j(j+1)$ (Abramowitz and Stegun, 1964: 828). For recent research on (1.7) see Mason (2001).

It is the purpose of this paper to consider the various expansions of sums and cubes suggested by the Fibonacci triangle and to extend the latter for even powers.

2. Algebraic Characterisation

With $N \in \mathbb{Z}_+$, the results in Table 1 can also be summarised by

$$N^3 = \sum_{t=t_0}^{t=t_m} (2t+1) \tag{2.1}$$

with $t_0 = \frac{1}{2}N(N-1)$ and $t_m = \frac{1}{2}N(N+1) - 1$. N may be further characterised by using the modular ring $\mathbb{Z}_4$ (Leyendekkers *et al*, 1997). Integers in this 4-class ring can be represented by $(4r_i + i)$ where $\bar{i}$ represents the class and r_i the row in $\bar{i}$ when the classes are set out as the columns in a rectangular array. Table 2 shows the class and parity structure of integers $N \in \mathbb{Z}_4$.

Class of N	Class of N^3	t_0 parity	t_0 class	t_m parity	t_m class	Parity of $r_i(N)$
$\bar{1}_4$	$\bar{1}_4$	even	$\bar{2}_4$	even	$\bar{2}_4$	odd
		even	$\bar{0}_4$	even	$\bar{0}_4$	even
$\bar{3}_4$	$\bar{3}_4$	odd	$\bar{3}_4$	odd	$\bar{1}_4$	even
		odd	$\bar{1}_4$	odd	$\bar{3}_4$	odd
$\bar{0}_4$	$\bar{0}_4$	even	$\bar{2}_4$	odd	$\bar{1}_4$	odd
		even	$\bar{0}_4$	odd	$\bar{3}_4$	even
$\bar{2}_4$	$\bar{0}_4$	odd	$\bar{1}_4$	even	$\bar{2}_4$	even
		odd	$\bar{3}_4$	even	$\bar{0}_4$	odd

Table 2: Class and parity characteristics from (2.1)

For example, if $N = 7$, the class is $\overline{3}_4$ and the row is 1, and hence odd. We expect t_0 to be odd and in Class $\overline{1}_4$ and t_m to be odd and in Class $\overline{3}_4$. From Equation (2.1), $t_0 = 21 = (4 \times 5 + 1)$ and so in $\overline{1}_4$, whilst $t_m = 27 = (4 \times 6 + 3)$ and so in $\overline{3}_4$.

As can be seen from Table 1, the even cubes, N^3, will be the sum of an even number (N) of odd numbers which are in $\overline{1}_4$ or $\overline{3}_4$, the classes alternating with each other; that is, $\overline{3}_4, \overline{1}_4, \overline{3}_4, \overline{1}_4, \overline{3}_4, \ldots$ Thus the sum can only be in $\overline{0}_4$. Class $\overline{2}_4$, in fact, contains no powers at all.

3. Difference of Squares

Another way of expressing the cubes is by a difference of squares:

$$N^3 = (\tfrac{1}{2}N)^2[(N+1)^2 - (N-1)^2]. \tag{3.1}$$

For even N we also have

$$N^3 = (\tfrac{1}{2}N)^2[(\tfrac{1}{2}(N+4))^2 - (\tfrac{1}{2}(N-4))^2]. \tag{3.2}$$

For convenience we use

$$N^3 = x^2 - y^2 \tag{3.3}$$

with the (x,y) pairs equal to $[\tfrac{1}{2}N(N+1), \tfrac{1}{2}N(N-1)]$ for all integers, and $[\tfrac{1}{4}N(N+4), \tfrac{1}{4}N(N-4)]$ for even integers as well.

Comparison with Equation (2.1) shows that the general (x,y) pair equals $[(t_m+1), t_0]$.

Table 3 lists some (x,y) pairs and the class structure of the integers within $\mathbb{Z}_4$. As can be seen, integers in $\overline{1}_4$ have an (x,y) pair structure of $(\overline{1}_4, \overline{0}_4)$ or $(\overline{3}_4, \overline{2}_4)$ which alternate with each other. For integers in $\overline{3}_4$ the (x,y) class structure is $(\overline{2}_4, \overline{3}_4)$ and $(\overline{0}_4, \overline{1}_4)$ which alternate with each other. For even integers, those in $\overline{2}_4$ have an (x,y) class structure of $(\overline{1}_4, \overline{3}_4), (\overline{3}_4, \overline{3}_4)$and $(\overline{3}_4, \overline{1}_4), (\overline{3}_4, \overline{3}_4)$ which alternate; all x,y in this case are odd. For integers in $\overline{0}_4$, the (x,y) sequence is $(\overline{2}_4, \overline{2}_4), (\overline{0}_4, \overline{0}_4)$ and $(\overline{0}_4, \overline{0}_4), (\overline{0}_4, \overline{0}_4)$ and all (x,y) pairs are even.

Note that $N = x - y$ in the general case and $N = \tfrac{1}{2}(x-y)$ for even N as well.

An interesting feature of Table 3 is that $(2x-1)$ or $(2t_m+1)$ of Equation (2.1) (that is, the last odd integer of the cubic sum) is commonly a prime (16 out of the 25 listed integers).

4. Extraction of Twos

If we expand Equation (2.1), then

$$N^3 = N + (N-1)2t_0 + 2t_m + 2\sum_{t=0}^{N-2} t. \tag{4.1}$$

Substituting for t_0, t_m from Section 1, we get

$$N^2 = 3N + q - 2 \tag{4.2}$$

in which

$$q = 2\sum_{t=0}^{N-2} t = (N-1)(N-2)$$

and

N	(x,y) $(x-y)=N$ $(x+y)^2=N^2, \forall N$	(x',y') $(x'-y')=2N$ $(x'+y')=N^2,\ 2\|N$	Class of N in $\mathbb{Z}_4$	Class of (x,y)	Class of (x',y')	Class of (x^2,y^2)	Class of (x'^2,y'^2)	Class of $x^2-y^2=N^3$
1	$(1,0)$		$\bar{1}$	$(\bar{1},\bar{0})$		$(\bar{1},\bar{0})$		$\bar{1}$
2	$(3,1)$		$\bar{2}$	$(\bar{3},\bar{1})$		$(\bar{1},\bar{1})$		$\bar{0}$
3	$(6,3)$		$\bar{3}$	$(\bar{2},\bar{3})$		$(\bar{0},\bar{1})$		$\bar{3}$
4	$(10,6)$	$(8,0)$	$\bar{0}$	$(\bar{2},\bar{2})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$\bar{0}$
5	$(15,10)$		$\bar{1}$	$(\bar{3},\bar{2})$		$(\bar{1},\bar{0})$		$\bar{1}$
6	$(21,15)$	$(15,3)$	$\bar{2}$	$(\bar{1},\bar{3})$	$(\bar{3},\bar{3})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$\bar{0}$
7	$(28,21)$		$\bar{3}$	$(\bar{0},\bar{1})$		$(\bar{0},\bar{1})$		$\bar{3}$
8	$(36,28)$	$(24,8)$	$\bar{0}$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$\bar{0}$
9	$(45,36)$		$\bar{1}$	$(\bar{1},\bar{0})$		$(\bar{1},\bar{0})$		$\bar{1}$
10	$(55,45)$	$(35,15)$	$\bar{2}$	$(\bar{3},\bar{1})$	$(\bar{3},\bar{3})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$\bar{0}$
11	$(66,55)$		$\bar{3}$	$(\bar{2},\bar{3})$		$(\bar{0},\bar{1})$		$\bar{3}$
12	$(78,66)$	$(48,24)$	$\bar{0}$	$(\bar{2},\bar{2})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$\bar{0}$
13	$(91,78)$		$\bar{1}$	$(\bar{3},\bar{2})$		$(\bar{1},\bar{0})$		$\bar{1}$
14	$(105,91)$	$(63,35)$	$\bar{2}$	$(\bar{1},\bar{3})$	$(\bar{3},\bar{3})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$\bar{0}$
15	$(120,105)$		$\bar{3}$	$(\bar{0},\bar{1})$		$(\bar{0},\bar{1})$		$\bar{3}$
16	$(136,120)$	$(80,48)$	$\bar{0}$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$\bar{0}$
17	$(153,136)$		$\bar{1}$	$(\bar{1},\bar{0})$		$(\bar{1},\bar{0})$		$\bar{1}$
18	$(171,153)$	$(99,63)$	$\bar{2}$	$(\bar{3},\bar{1})$	$(\bar{3},\bar{3})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$\bar{0}$
19	$(190,171)$		$\bar{3}$	$(\bar{2},\bar{3})$		$(\bar{0},\bar{1})$		$\bar{3}$
20	$(210,190)$	$(120,80)$	$\bar{0}$	$(\bar{2},\bar{2})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$\bar{0}$
21	$(231,210)$		$\bar{1}$	$(\bar{3},\bar{2})$		$(\bar{1},\bar{0})$		$\bar{1}$
22	$(253,231)$	$(143,99)$	$\bar{2}$	$(\bar{1},\bar{3})$	$(\bar{3},\bar{3})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$\bar{0}$
23	$(276,253)$		$\bar{3}$	$(\bar{0},\bar{1})$		$(\bar{0},\bar{1})$		$\bar{3}$
24	$(300,276)$	$(168,120)$	$\bar{0}$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$\bar{0}$
25	$(325,300)$		$\bar{1}$	$(\bar{1},\bar{0})$		$(\bar{1},\bar{0})$		$\bar{1}$

Table 3: Class structure within $\mathbb{Z}_4$

$$N = \tfrac{1}{2}(3+Q),\ \ N > 1, \qquad (4.3)$$
$$N = \tfrac{1}{2}(3-Q),\ \ N = 1, \qquad (4.4)$$

with

$$Q = (1 + 4q)^{½}.$$

From Equation (4.3) with $N > 1$

$$N^2 = ½(5 + 2q + 3Q) \tag{4.5}$$

and

$$N^3 = ¼(5 + 2q + 3Q)(3 + Q) \tag{4.6}$$

or

$$N^3 = ¼[18(1 + q) + (14 + 2q)Q] \tag{4.7}$$

or

$$N^3 = \frac{1}{8}Q(Q^2 + 9Q + 27) + 27. \tag{4.8}$$

For $N = 1$, use $-Q$. Combining Equation (3.3) with $(x,y) = ((t_m + 1), t_0)$ and Equation (4.1) gives

$$N = (N - 1)^2 - (q - 1). \tag{4.9}$$

Thus we have N^3 as a sum of consecutive odd numbers, a difference of squares or a function of q, whilst any integer or integer squared may also be expressed as a function of q

It is of interest that if Q is a prime, then $3|q$. One third of the first hundred integers have $3 \nmid q$, and 43% have Q as a prime. Half of the latter integers are odd with 75% of these being prime (exceptions being 25, 35, 55, 65, 91).

Both q and Q can be further characterized by using the right end digits (REDs), indicated by an asterisk in Table 4. The correct q^* and Q^* can be ascertained by checking equations for compatibility. Thence $f(q)$ and $f(Q)$ can be used to simplify the equation.

q^*	Q^*	$f(q)$	$f(Q)$
0	9	$q = 25w^2 + 45w + 20$	$Q = 9 + 10w$
	1	$q = 25w^2 + 5w + 0$	$Q = 1 + 10w$
2	7	$q = 25w^2 + 35w + 12$	$Q = 7 + 10w$
	3	$q = 25w^2 + 15w + 2$	$Q = 3 + 10w$
6	5	$q = 25w^2 + 25w + 6$	$Q = 5 + 10w$

Table 4: Right end digits for q, Q

Note that the right-end digits for $(q : Q)$ are only $(0 : 1,9), (2 : 3,7)$ and $(6 : 5)$.

5. Expansion of Squares

The sums of odd numbers which make up squares are also sums of odd integers:

$$2^2 = 1+3,$$
$$3^2 = 1+3+5,$$
$$4^2 = 1+3+5+7,$$

and so on as in Table 5 in which the row sums equals the outer diagonal sum.

1^2	1					
2^2	1	3				
3^2	1	3	5			
4^2	1	3	5	7		
5^2	1	3	5	7	9	
6^2	1	3	5	7	9	11

Table 5: Table of squares

Thus

$$N^2 = \sum_{t=0}^{N-1}(2t+1)$$
$$= N + 2\sum_{t=0}^{N-1} t. \tag{5.1}$$

This can be compared with the extraction of twos for cubes. For example, Equation (4.2) gives

$$N^2 = (3N-2) + 2\sum_{t=0}^{N-2} t, \tag{5.2}$$

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or, on squaring Equation (4.3):

$$N^2 = \tfrac{1}{2}(5 + 2q + 3(1+4q)^{\frac{1}{2}}) \tag{5.3}$$

with $q = 2\sum_{t=0}^{N-2} t$. For all N with $N^3 = x^2 - y^2$,

$$N^2 = x + y = \frac{N^3}{x-y}. \tag{5.4}$$

As well, for even N with $N^3 = x'^2 - y'^2$,

$$N^2 = 2(x' + y') = \frac{2N^3}{x' - y'} \tag{5.5}$$

with

$$(x,y) = (\tfrac{1}{2}N(N+1), \tfrac{1}{2}N(N-1))$$

and

$$(x',y') = (\tfrac{1}{4}N(N+4), \tfrac{1}{4}N(N-4));$$

see Section 1. From Equation (4.9),

$$N^2 = N^4 - 4N^3 + (8-2q)N^2 + 4(q-2)N + (q-2)^2. \qquad (5.6)$$

6. Examples

We now give some examples of usage of the foregoing.

(1) The q functions may be used to solve a cubic, for instance. Let

$$x^3 - 12x + 9 = 0. \qquad (6.1)$$

Substituting in Equations (4.4), (4.7) and (4.8), we get

$$4q = \left(\frac{81(q-1)^2}{(5-q)^2}\right) - 1 \qquad (6.2)$$

which gives $q = 2$; and since$q = (x-1)(x-2)$ from Equation (4.2), then $x = 3$ as the only integer solution. Alternatively, we can use

$$q = \left(\frac{7(9q-2)}{30-q}\right)^{1/2} \qquad (6.3)$$

which again yields $q = 2$.

(2) Consider the elliptic function

$$y^2 = x^3 + Ax^2 + x. \qquad (6.4)$$

Obviously there will only be integer solutions when $A = 2$ and then $y = (x+1)x^{1/2}$.
Using $x = \tfrac{1}{2}(3+Q)$ and $y = \tfrac{1}{2}(3+Q')$ from Equation (4.3), we get, with $A = 2$,

$$Q' = -3 + \tfrac{1}{2}(Q+5)(6+2Q)^{1/2}. \qquad (6.5)$$

Thus, with $Q = 5, Q' = 17$ and this gives $y = 10$ and $x = 4$.
In general,

$$2(3+Q')^2 = (3+Q)(Q^2 + (6+2A)Q + (13+6A)) \qquad (6.6)$$

with $(13+6A) = ab$ and $(6+2A) = a+b$,

$$b = (A+3) + (A^2-4)^{1/2}. \qquad (6.7)$$

When $A = 2, b = 5$ and $a = 5$, otherwise there are no integer solutions for b as $A^2 - 4$ can never be a square.

(3) Consider the Pythagorean triple

$$c^2 = a^2 + b^2. \qquad (6.8)$$

Substituting in Equation (5.1) we obtain

$$c^2 - b^2 = (c-b) + 2\sum_{t=b}^{c-1} t, \tag{6.9}$$

and with $z = c - b$

$$c^2 - b^2 = z + 2\sum_{t=b}^{b+z-1} t; \tag{6.10}$$

with $z = 1$ (for triples $(5,4,3), (25,24,7), (221,220,21)$ for example)

$$c^2 - b^2 = 1 + 2b; \tag{6.11}$$

whilst, with $z = 2$ (for triples $(17,15,8), (10405,10403,204)$ for example)

$$c^2 - b^2 = 2 + 2\sum_{t=b}^{b+1} t = 4 + 4b, \tag{6.12}$$

and so on.

(4) From Equation (5.1)

$$N^3 = N^2 + 2N\sum_{t=0}^{N-1} t. \tag{6.13}$$

If we have

$$c^3 = a^3 + b^3 + d^3, \tag{6.14}$$

then

$$c^3 - b^3 = c^2 - b^2 + 2\left[c\sum_{t=0}^{c-1} t - b\sum_{t=0}^{b-1} t \right], \tag{6.15}$$

and

$$a^3 + d^3 = a^2 + d^2 + 2\left[a\sum_{t=0}^{a-1} t + d\sum_{t=0}^{d-1} t. \right] \tag{6.16}$$

This symmetry indicates that integer solutions will occur, unlike the asymmetric triple form. With $c = 6$, $a = 5$, $b = 4$ and $d = 3$, the right hand side of Equation (6.15) equals 91 and the right hand side of Equation (6.16) equals 91.

In general, if we take

$$n^3 + (n+1)^3 + (n+2)^3 = (n+3)^3,$$

as for the $n = 3$ case, and using $n^3 = x^2 - y^2$, and noting that for those cases (such as $n = 3$ above), we have x for $(n+3) = x$ for $(n+2)$. Equating these shows that $n = 3$ is the only solution in such cases.

(5) Equation (5.6) gives N^4 as a function of N and q. Thus a polynomial of the fourth degree may be reduced to a cubic or lower. For instance, consider

$$x^4 - 8x^3 + 24x^2 - 32x + 15 = 0 \tag{6.17}$$

which is known to have two complex roots and two real ones. Substituting for x^4 from Equation (5.6) yields

$$-4x^3 + (17 + 2q)x^2 - (24 + 4q)x + 15 - (q - 2)^2 = 0. \tag{6.18}$$

With $q = 0, x = 1$ or 2, and substitution of q shows $x = 1$. With $q = 2, x = 3$, which satisfies the equation. Thus, the real roots are $x = 1, 3$.

For cases where $q > 0$, that is $x \neq 1, 2$, we can substitute Equations (4.2) and (4.7) into Equation (6.18) to obtain

$$x = \frac{(14 + 2q)Q - q(q - 1) + 41}{27 + 2q}; \tag{6.19}$$

when $q = 2, Q = 3$ so that $x = 3$ as before.

(6) Given that

$$y = 8x^3 - 36x^2 + 56x - 39,$$

find the values of x which will make y a perfect cube.
Substitution in Equation (4.3), (4.5) and (4.7) for x, x^2, x^3 respectively gives:

$$y = Q^3 + Q - 9. \tag{6.20}$$

If $Q = 9, q = 20$ and since $q = 2\sum_{t=0}^{x-2} t$, then $x = 6$. Thus $x = 6$ makes $y = Q^3 = 9^3$.

(7) Show that $5^{2n} - 1$ is always divisible by 24.
This is easily solved from $\mathbb{Z}_4$. Since $3 \nmid 5^n$ and $(5^n)^2 \in \bar{1}_4$ since it is a square, then

$$(5^n)^2 - 1 = 4r_1$$

where, as before, r_1 is the row. Since $(5^n, 3) = 1$, all other r_1 for squares have $6|r_1$ so that $4r_1$ always has 24 as a factor.

7. Conclusion

By way of conclusion we note that Fibonacci's odd number triangle for cubes can be extended to an odd number triangle for fourth powers as in Table 6.

1^4	$+0$	1								
2^4	$+1^4$	3	5	7						
3^4	$+2^4$	9	11	13	15	17				
4^4	$+3^4$	19	21	23	25	27	29	31		
5^4	$+4^4$	33	35	37	39	41	43	45	47	49

Table 6: Triangle for fourth powers

We observe that

$$N^4 = (N-1)^4 + \sum_{j=(N-1)^2}^{N^2-1} (2j+1), \ \ n > 1,$$

and

$$\sum_{j=1}^{N} k^j = \sum_{j=1}^{N} (k-1)^j + \sum_{j=1}^{N^2-1} (2j-1),$$

by comparison with the results in Section 1. An alternative triangle for odd-integer cubes can also be provided from

$$N^3 = \sum_{j=A}^{B} j \tag{7.1}$$

with $A = 0$ for $N = 1$ and for $N \geq 3$

$$A = 2\left(1 + \sum_{j=1}^{½(N-3)} (2j+1)\right),$$
$$B = (2N-1) + A.$$

1^3	1													
3^3	2	3	4	5	6	7								
5^3	8	9	10	11	12	13	14	15	16	17				
7^3	18	19	20	21	22	23	24	25	26	27	28	29	30	31

Table 7: Triangle for odd integer cubes

Thus in general

$$N^n = \sum_{j=A}^{B} jN^{n-3}. \tag{7.2}$$

Equation (7.2) can be readily established by induction on n, for $n \geq 3$, since it can be readily seen that Equation (7.1) is the case of (7.2) when $n = 3$.

The basis of many efficient primality tests and nearly all composite tests is Fermat's Little Theorem (Riesel, 1994), namely, that if p is prime and $(a,p) = 1$, then

$$a^{p-1} \equiv 1(\text{mod} p).$$

This can be used as a test of compositeness: namely, N is composite if $(a,N) = 1$, and

$$a^{N-1} \not\equiv 1(\text{mod} N).$$

Since $N - 1$ is even, we can use

$$a^{N-1} = 1 + \sum_{j=1}^{a^{½(N-1)}-1} (2j+1)$$

$$\not\equiv 1(\mathrm{mod}\,N).$$

For an odd power, n, since

$$N^{2n} = \sum_{j=0}^{N^n-1} (2j+1),$$

so that

$$N^n = \left[\sum_{j=0}^{N^n-1} (2j+1) \right]^{½}.$$

It is of interest in the context of this paper that many primality tests also involve generalisations of the Fibonacci numbers (Müller, 2000).

In general, for triangles confined to the sum of odd integers, even powers, n follow:

$$N^n = \sum_{j=0}^{N^{½n}-1} (2j+1)$$

$$= N^{½n} + 2 \sum_{j=0}^{N^{½n}-1} j$$

with the number of odd integers in each row equal to $N^{½n}$. When $n > 2$ the row sums will no longer equal the outer diagonal sums. With $N > M$ and n even, if

$$N^n + M^n = 2 \sum_{j=0}^{M^{½n}-1} (2j+1) + \sum_{j=M^{½n}}^{N^{½n}-1} (2j+1)$$

$$= P^n,$$

then $P \in \mathbb{Z}$ iff $n = 2$ according to Fermat's Last Theorem. That is,

$$\sum_{j=0}^{M^{½n}-1} (2j+1) \neq \sum_{j=N^{½n}}^{P^{½n}-1} (2j+1)$$

unless

$$\sum_{j=0}^{M-1}(2j+1) = \sum_{j=N}^{P-1}(2j+1)$$

which could be related to the class constraints for powers of even and odd integers within $\mathbb{Z}_4$ ($M^{½n}$, $N^{½n}$, $P^{½n}$). Even powers are confined to one class which reduces the probability of a match. Odd powers also have class constraints (Table 2).

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