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On Heron Triangles, III

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1. Let ABC be a triangle with lengths of sides $BC = a$, $AC = b$, $AB = c$ positive integers. Then ABC is called a Heron triangle (or simply, H-triangle) if its area $\Delta = Area(ABC)$ is an integer number. The theory of H-triangles has a long history and certain results are many times rediscovered. On the other hand there appear always some new questions in this theory, or even there are famous unsolved problems. It is enough (see e.g. [2]) to mention the difficult unsolved problem on the existence of a H-triangle having **all** medians integers. The simplest H-triangle is the Pythagorean triangle (or P-triangle, in what follows). Indeed, by supposing AB as hypothenuse, the general solution of the equation

$$a^2 + b^2 = c^2 \tag{1}$$

(i.e. the so-called **Pythagorean numbers**) are given by

$$a = \lambda(m^2 - n^2), \quad b = 2\lambda mn, c = \lambda(m^2 + n^2) \quad \text{(if } b \text{ is even)} \tag{2}$$

where λ is arbitrary positive integer, while $m > n$ are coprime of different parities (i.e. $(m, n) = 1$ and m and n cannot be both odd or even). Clearly $\Delta = \dfrac{ab}{2} = \lambda^2 mn(m^2 - n^2)$, integer.

Let p be the **semiperimeter** of the triangle. From (2) $p = \lambda(m^2 + mn)$; and denoting by r the **inradius** of a such triangle, it is well known that

$$r = p - c \tag{3}$$

implying that r is always integer.

On the other hand, the radius R of the **circumscribed circle** in this case is given by the simple formula

$$R = \frac{c}{2} \tag{4}$$

which, in view of (2) is integer only if λ is even, $\lambda = 2\Lambda$ ($\Lambda > 0$). The **heights** of a P-triangle are given by

$$h_a = b, \quad h_a = b, \quad h_c = \frac{ab}{c}; \tag{5}$$

therefore **all** heights are integers only if $c|ab$, which, by (2) can be written as $(m^2 + n^2)|2\lambda mn(m^2 - n^2)$. Since $(m, n) = 1$, of different parity, it is immediate that $(m^2 + n^2, 2mn(m^2 - n^2)) = 1$, giving $(m^2 + n^2)|\lambda$; i.e. $\lambda = K(m^2 + n^2)$ ($K > 0$).

By summing, in a P-triangle the following elements: $\Delta, h_a, h_b, h_c, r, R$ are integers at the same time if and only if a, b, c are given by

$$a = 2d(m^4 - n^4), \quad b = 4dmn(m^2 + n^2), \quad c = 2d(m^2 + n^2)^2, \tag{6}$$

where we have denoted $K = 2d$ (as by (4), λ is even and $m^2 + n^2$ is odd).

In fact this contains the particular case of the P-triangles with $a = 30n$, $b = 40n$, $c = 50n$ in a problem [7] by F. Smarandache, and in fact gives **all** such triangles.

2. An interesting example of a H-triangle is that which has as sides **consecutive integers**. Let us denote by CH such a H-triangle (i.e. "consecutive Heron"). The CH-triangles appear also in the second part [6] of this series, where it is proved that r is always integer. Since in a H-triangle p is always integer (see e.g. [3], [4]), if $x - 1, x, x + 1$ are the sides of a CH-triangle, by $p = \frac{3x}{2}$, we have that x is even, $x = 2y$. Therefore the sides are $2y - 1, 2y, 2y + 1$, when $p = 3y$, $p - a = y + 1$, $p - b = y$, $p - c = y - 1$ giving $\Delta = \sqrt{3y(y+1)(y-1)} = y\sqrt{3(y^2 - 1)}$, by Heron's formula of area. This gives $\frac{\Delta}{y} = \sqrt{3(y^2 - 1)}$ = rational. Since $3(y^2 - 1)$ is integer, it must be a perfect square,

$3(y^2-1)=t^2$, where

$$\Delta = yt. \tag{7}$$

Since the prime 3 divides t^2, clearly $3|t$, let $t=3u$. This implies

$$y^2 - 3u^2 = 1 \tag{8}$$

which is a "Pell-equation". Here $\sqrt{3}$ is an irrational number, and the theory of such equations (see e.g. [5]) is well-known. Since $(y_1, u_1) = (2,1)$ is a basic solution (i.e. with y_1 the smallest), **all** other solutions of this equations are provided by

$$y_n + u_n\sqrt{3} = \left(2+\sqrt{3}\right)^n \quad (n \geq 1). \tag{9}$$

By writting

$$y_{n+1} + u_{n+1}\sqrt{3} = \left(2+\sqrt{3}\right)^{n+1} = \left(y_n + u_n\sqrt{3}\right)\left(2+\sqrt{3}\right) = 2y_n + 3u_n + \sqrt{3}(y_n + 2u_n),$$

we get the recurrence relations

$$\begin{cases} y_{n+1} = 2y_n + 3u_n \\ u_{n+1} = y_n + 2u_n \end{cases} \quad (n = 1, 2, \ldots) \tag{10}$$

which give all solution of (8); i.e. all CH-triangles (**all** such triangles have as sides $2y_n - 1$, $2y_n$, $2y_n + 1$). By $y_n = 2, 7, 26, 97, \ldots$ we get the CH-triangles $(3,4,5)$; $(13,14,15)$; $(51,52,53)$; $(193,194,195)$; $\ldots$.

Now, we study certain particular elements of a CH-triangle. As we have remarked, r is **always** integer, since

$$r = \frac{\Delta}{p} = \frac{\Delta}{3y} = \frac{t}{3} = u$$

(in other words, in (10) u_n represents the inradius of the nth CH-triangle). If one denotes by h_{2y} the **height** corresponding to the (single) even side of this triangle, clearly

$$h_{2y} = \frac{2\Delta}{2y} = \frac{\Delta}{y} = 3r.$$

Therefore we have the interesting fact that h_{2y} is integer, and even more, r is the **third part** of this height. On the other hand, in a CH-triangle, which is **not** a P-triangle (i.e. **excluding** the triangle $(3,4,5)$), all other heights cannot be integers. (11)

Indeed, $\frac{(2y-1)x}{2} = \Delta = yt$ gives $(2y-1)x = 2yt$ (here $x = h_{2y-1}$ for simplicity). Since $(u,y) = 1$ and $t = 3u$ we have $(t,y) = 1$, so $x = \frac{2yt}{2y-1}$ is integer only if $(2y-1)|t = 3u$. Now, by $y^2 - 3u^2 = 1$ we get $4y^2 - 1 = 12u^2 + 3$, i.e. $(2y-1)(2y+1) = 3(4u^2+1) = 4(3u^2) \nmid 3$. Therefore $(2y-1)|3u$ implies $(2y-1)|3u^2$, so we must have $(2y-1)|3$, implying $y = 2$ ($y > 1$). For h_{2y+1} we have similarly $\frac{(2y+1)z}{2} = \Delta = yt$, so $z = \frac{2yt}{2y+1}$, where $(2y+1)|t = 3u \ \Leftrightarrow \ (2y+1)|3 \ \Leftrightarrow \ y = 1$ (as above). Therefore $z = h_{2y+1}$ **cannot** be integer in **all** CH-triangles. (Remember that $x = h_{2y-1}$ is integer **only** in the P-triangle $(3,4,5)$).

For R the things are immediate:

$$R = \frac{abc}{4\Delta} = \frac{2y(4y^2-1)}{4yt} = \frac{4y^2-1}{2t} = \frac{\text{odd}}{\text{even}} \neq \text{integer}. \tag{12}$$

Let now r_a denote the radius of the **exscribed** circle corresponding to the side of length a. It is well-known that

$$r_a = \frac{\Delta}{p-a}.$$

By $r_{2y} = \frac{yt}{y}$ ($= 3r$, in fact), we get that r_{2y} is **integer**. Now $r_{2y-1} = \frac{yt}{y+1}$, $r_{2y+1} = \frac{yt}{y-1}$. Here $(y+1, y) = 1$, so r_{2y-1} is integer only when $(y+1)|t = 3u$. Since $y^2 - 3u^2 = 1$ implies $(y-1)(y+1) = 3u^2 = u(3u)$, by $3u = (y+1)k$ one has $3(y-1) = 3uk = (y+1)k^2$ and $y - 1 = uk$. By $k^2 = \frac{3(y-1)}{y+1} = 3 - \frac{6}{y+1}$ we get that $(y+1)|6$, i.e. $y \in \{1,2,5\}$. We can have only $y = 2$, when $k = 1$.

Therefore r_{2y-1} **is integer only in the P-triangle** $(3,4,5)$. (13)

In this case (and only this) $r_{2y+1} = \frac{2 \cdot 3}{2-1} = 6$ is integer, too.

Remarks 1. As we have shown, in all CH-triangle, which is not a P-triangle, we can exactly one height, which is integer. Such triangles are all acute-angled. (Since $(2y-1)^2 +$

$(2y)^2 > (2y+1)^2$). In [4] it is stated as an open question if in all acute-angled H-triangles there exists **at least** an integer (-valued) height. This is not true, as can be seen from the example $a = 35$, $b = 34$, $c = 15$. (Here $34^2 + 15^2 = 1156 + 225 = 1381 > 35^2 = 1225$, so ABC is acute-angled). Now $p = 42$, $p - a = 7$, $p - b = 8$, $p - c = 27$, $\Delta = 252 = 2^2 \cdot 3^2 \cdot 7$ and $35 = 7 \cdot 5 \nmid 2\Delta$, $34 = 2 \cdot 17 \nmid 2\Delta$, $15 = 3 \cdot 5 \nmid 2\Delta$. We note that $h_a = \dfrac{2\Delta}{a}$ is integer only when a divides 2Δ. Let n be an integer such that $5 \cdot 17 \nmid n$. Then $35n, 34n, 15n$ are the sides of a H-triangle, which is acute-angled, and no height is integer. The H-triangle of sides $39, 35, 10$ is **obtuse-angled**, and no height is integer.

3. Let now ABC be an **isosceles** triangle with $AB = AC = b$, $BC = a$. Assuming that the heights $AA' = x$ and $BB' = y$ are integers (clearly, the third height $CC' = BB'$), by $b^2 = x^2 + \left(\dfrac{a}{2}\right)^2$ we have $\dfrac{a^2}{4} = b^2 - x^2 =$ integer, implying a=even. Let $a = 2u$. Thus

$$b^2 = x^2 + u^2. \tag{14}$$

We note that if x is integer, then $a = 2u$, so ABC is a H-triangle, since $\Delta = \dfrac{xa}{2} = xu$. The general solutions of (14) (see (2)) can be written as one of the followings:

(i) $b = \lambda(m^2 + n^2)$, $x = \lambda(m^2 - n^2)$, $u = 2\lambda mn$;

(ii) $b = \lambda(m^2 + n^2)$, $x = 2\lambda mn$, $u = \lambda(m^2 - n^2)$.

We shall consider only the case (i), the case (ii) can be studied in a completely analogous way.

Now $a = 4\lambda mn$, $b = \lambda(m^2 + n^2)$; so $\Delta = 2\lambda^2 mn(m^2 - n^2)$. Thus $y = \dfrac{2\Delta}{b}$ is integer only when $\lambda(m^2 + n^2) | 4\lambda^2 mn(m^2 - n^2)$. Thus $y = \dfrac{2\Delta}{b}$ is integer only when $\lambda(m^2 + n^2) | 4\lambda^2 mn(m^2 - n^2)$. Since $(m^2 + n^2, 4mn(m^2 - n^2)) = 1$ (see **1.**, where the case of P-triangles has been considered), this is possible only when $(m^2 + n^2) | \lambda$, i.e.

$$\lambda = s(m^2 + n^2). \tag{15}$$

Therefore, in an **isosceles H-triangle**, having **all heights** integers, we must have (in case (ii) $a = 4smn(m^2 + n^2)$; $b = s(m^2 + n^2)^2$ (where a is the **base** of the triangle) or (in case (ii))

$$a = 2sm(m^4 - n^4), \quad b = sm(m^2 + n^2)^2. \tag{16}$$

We note here that case (ii) can be studied similarly to the case (i) and we omit the details.

In fact, if an isosceles triangle ABC with integer sides a, b (base a) is H-triangle, then $p = b + \frac{a}{2} =$ integer, so $a = 2u =$ even. So $p = b + u$ and $p - b = u$, $p - a = b - \frac{a}{2} = b - u$, implying $\Delta = \sqrt{p(p-a)(p-b)^2} = u\sqrt{b^2 - u^2}$. This is integer only when $b^2 - u^2 = q^2$, when $\Delta = uq$. Now $b^2 - u^2$ is in fact x^2 (where x is the height corresponding to the base a), so $q = x$. In other words, if an isosceles triangles ABC is H-triangle, then its **height x must be integer**, and we recapture relation (14). Therefore, in an isosceles H-triangle a height is always integer (but the other ones only in case (16)). In such a triangle, $r = \frac{\Delta}{p} = \frac{uq}{b+u}$, where $b^2 = u^2 + q^2$. By (2) we can write the following equations:

i) $b = \lambda(m^2 + n^2)$, $u = 2\lambda mn$, $q = \lambda(m^2 - n^2)$;

ii) $b = \lambda(m^2 + n^2)$, $u = \lambda(m^2 - n^2)$, $q = 2\lambda mn$.

In case i) $b + u = \lambda(m+n)^2 | uq = 2\lambda^2 mn(m^2 - n^2)$ only iff $(m+n)^2 | 2\lambda mn(m^2 - n^2)$, i.e. $(m+n) | 2\lambda mn(m-n)$; and since $(m+n, 2mn(m-n)) = 1$. This is possible only when $(m+n)|\lambda$, i.e.

$$\left.\begin{array}{l} \lambda = s(m+n) \\ \text{case (ii) we get } m|\lambda, \text{ so } \lambda = sm \end{array}\right\} \tag{17}$$

Therefore in an isosceles triangle r **is integer** only when

i) $b = s(m+n)(m^2 + n^2)$, $a = 2n = 4mns(m+n)$; or

ii) $b = sm(m^2 + n^2)$, $a = sm(m^2 - n^2)$.

For $R = \frac{abc}{4\Delta} = \frac{ab^2}{4\Delta} = \frac{2nb^2}{4nq} = \frac{b^2}{2q}$ we have that R is integer only when $2q|b^2$, where $b^2 = n^2 + q^2$. In case i) we get $2\lambda(m^2 - n^2)|\lambda^2(m^2 + n^2)^2$, which is possible only when $2(m^2 - n^2)|\lambda$ or in case ii) $4\lambda mn|\lambda^2(m^2 + n^2)^2$ i.e. $4mn|\lambda$. By summing, R is integer only if in i) $\lambda = 2s(m^2 - n^2)$, while in ii), $\lambda = 4smn$. Then the corresponding sides a, b can be written explicitely.

From the above considerations we can determine **all** isosceles H-triangles, in which **all heights** and r, R are integers. These are one of the following two cases:

$$\begin{aligned} &1)\quad a = 4kmn(m^4 - n^4),\quad b = 2k(m^2 - n^2)(m^2 + n^2)^2;\\ &2)\quad a = 4kmn(m^4 - n^4),\quad b = 2kmn(m^2 + n^2) \end{aligned} \tag{18}$$

where $k \geq 1$ is arbitrary and $(m, n) = 1$, $m > n$ are of different parity.

In the same manner, by $r_a = \dfrac{\Delta}{p - a} = \dfrac{uq}{b - u}$ in case i) $b - u = \lambda(m - n)^2 | uq = 2\lambda^2 mn(m^2 - n^2)$ only if $(m - n)|\lambda$ i.e. $\lambda = s(m - n)$, while in case ii) $b - u = 2\lambda n^2 | uq = 2\lambda^2 mn(m^2 - n^2)$ iff $n|\lambda$, i.e. $\lambda = sn$. We can say that r_a is integer only if $\lambda = s(m - n)$ in i) and $\lambda = sn$ in ii). We omit the further details.

4. As we have seen in Remarks 1 there are infinitely many H-triangles having none of its heights integers (though, they are of course, rationals). Clearly, if at least a height of an **integral triangle** (i.e. whose sides are all integers) is integer, or rational its area is rational. We now prove that in this case the triangle is Heron. More precisely if a height of an integral triangle is rational, then this is a H-triangle. Indeed, by $\Delta = \dfrac{xa}{2}$ = rational, we have that Δ is rational.

On the other hand, by Heron's formula we easily can deduce

$$16\Delta^2 = 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4). \tag{19}$$

Therefore $(4\Delta)^2$ is integer. Since Δ = rational, we must have 4Δ = integer. If we can prove that $4|4\Delta$ then clearly Δ will be integer. For this it is sufficient to show $(4\Delta)^2 = 8k$ (since, then $4\Delta = 2l$ so $(4\Delta)^2 = 4l^2\vdots$ implies l = even). It is an aritmetic fact that $2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$ is **always** divisible by 8 (which uses that for x odd $x^2 \equiv 1(\mathrm{mod}4)$, while for x even, $x^2 \equiv 0(\mathrm{mod}4)$).

Let now ABC be a H-triangle with $BC = a$ = odd. We prove that the height

$$AA' = h_a \text{ is integer only if } a|(b^2 - c^2). \tag{20}$$

Indeed, let Δ be integer, with a, b, c integers. Then h_a is integer iff $a|2\Delta$. But this is equivalent to $a^2|4\Delta^2$ or $4a^2|16\Delta^2$. Now, by (19)

$$4a^2|[2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)] \Leftrightarrow a^2|(2b^2c^2-b^4-c^4) = -(b^2-c^2)^2$$

(since the paranthesis in bracket is divisible by 8 and $(a^2, 4) = 1$). Or, $a^2|(b^2-c^2)^2$ is equivalent to $a|(b^2-c^2)$.

Clearly, (19) implies $a^2|(b^2-c^2)^2$ for all a, therefore if

$$a \nmid (b^2-c^2) \tag{21}$$

h_a **cannot be** integer. But (19) is **not equivalent** with (20) for all a (especially, for $a =$ even). In fact (19) is the **exact** condition on the integrality of h_a in a H-triangle.

For general H-triangle, the conditions on the integrality of heights on r, R are not so simple as shown in the preceeding examples of P, CH or isosceles H-triangles.

In fact, from (19) we can remark that, since for $a =$ even we have $4a^2|a^4$ and $4a^2|a^2.(b^2+c^2)$ (by $a+b+c =$ even in a H-triangle), we can state that in a H-triangle h_a is integer iff:

$$\begin{cases} a|(b^2-c^2), & \text{for } a = \text{odd} \\ 2a|(b^2-c^2), & \text{for } a = \text{even} \end{cases}$$

Sometimes we can give simple negative results of type (21). One of these is the following:

Suppose that in an integral triangle of sides a, b, c we have

$$2(a+b+c) \nmid abc. \tag{22}$$

Then r, R **cannot be both** integers.

Indeed, suppose a, b, c, r, R integers. Since $r = \dfrac{\Delta}{p}$, clearly Δ is rational, so by the above argument, Δ is integer. So p is integer, too, $\dfrac{a+b+c}{2}|\Delta \Leftrightarrow a+b+c|2\Delta$. Now $R = \dfrac{abc}{4\Delta}$, so $4\Delta|abc$. Therefore $2(a+b+c)|4\Delta|abc$ if all the above are satisfied. But this is impossible, by assumption.

Certain direct results follow from the elementary connections existing between the elements of a triangle.

For example, from $R = \dfrac{b}{2\sin B}$ and $\sin B = \dfrac{h_a}{c}$ we get $R = \dfrac{bc}{2h_a}$, implying the following assertion:

If in an integral triangle of sides a, b, c we have $h_a =$ integer, then R is integer only if

$$2h_a | bc. \tag{23}$$

This easily implies the following negative result:

If in an integral triangle of sides a, b, c all heights h_a, h_b, h_c are integers, but one of a, b, c is not even; then R cannot be integer.

Indeed, by (23) $2h_a|bc$, $2h_b|ac$, $2h_c|ab$ so bc, ac, ab are all even numbers. Since $a+b+c = 2p$ is even, clearly all of a, b, c must be even.

5. The characterization of the above general problems (related to an arbitrary H-triangle) can be done if one can give general formulae for the most general case. Such formulae for a H-triangle have been suggested by R.D. Carmichael [1], and variants were many times rediscovered. We wish to note on advance that usually such general formulae are quite difficult to handle and apply in particular cases because the many parameters involved. The theorem by Carmichael can be stated as follows:

An integral triangle of sides a, b, c is a H-triangle if and only if a, b, c can be represented in the following forms

$$a = \frac{(m-n)(k^2+mn)}{d}, \quad b = \frac{m(k^2+n^2)}{d}, \quad c = \frac{n(k^2+m^2)}{d} \tag{24}$$

where d, m, n, k are positive integers; $m > n$; and d is an arbitrary common divisor of $(m-n)(k^2+mn)$, $m(k^2+n^2)$, $n(k^2+m^2)$.

For a complete proof we quote [3].

Now, from (24) we can calculate $p = \dfrac{m(k^2+mn)}{d}$ and

$$\Delta = \frac{kmn(m-n)(k^2+mn)}{d^2}.$$

In fact, the proof of (24) involves that p and Δ are integers for all k, m, n, d as given above. By simple transformations, we get

$$h_a = \frac{2\Delta}{a} = \frac{2kmn}{d}, \quad h_b = \frac{2\Delta}{b} = \frac{2kn(m-n)(k^2+mn)}{d(k^2+n^2)},$$

$$h_c = \frac{2kmn(m-n)(k^2+mn)}{d(k^2+m^2)},$$

$$r = \frac{\Delta}{p} = \frac{kn(m-n)}{d}, \quad R = \frac{abc}{4\Delta} = \frac{(k^2+m^2)(k^2+n^2)}{4kd}. \tag{25}$$

These relations enables us to deduce various conditions on the integer values of the above elements.

Particularly, we mention the following theorem:

All integral triangles of sides a, b, c which are H-triangles, and where r is integer are given by formulae (24), where d is **any common divisor** of the following expressions:

$$(m-n)(k^2+mn); \quad m(k^2+n^2); \quad n(k^2+m^2); \quad kn(m-n). \tag{26}$$

6. As we have considered before, among the CH-triangles in which **all** of r, r_a, r_b, r_c are integers are in fact the P-triangles.

In what follows we will determine all H-triangles having r, r_a, r_b, r_c integers.

Therefore, let

$$r = \frac{\Delta}{p} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}, \quad r_a = \frac{\Delta}{p-a} = \sqrt{(p(p-b)(p-c)},$$

$$r_b = \frac{\Delta}{p-b} = \sqrt{p(p-a)(p-c)}, \quad r_c = \sqrt{p(p-a)(p-b)}$$

be integers.

Put $p-a = x$, $p-b = y$, $p-c = z$, when $3p - 2p = x + y + z = p$.

Then $\sqrt{yz(x+y+z)}$, $\sqrt{xz(x+y+z)}$, $\sqrt{xy(x+y+z)}$, $\sqrt{xyz/(x+y+z)}$ are all integers, and since x, y, z are integer, the expressions on radicals must be perfect squares of integers. Let

$$xy(x+y+z) = t^2, \quad xz(x+y+z) = p^2, \quad yz(x+y+z) = q^2, \quad \frac{xyz}{x+y+z} = u^2. \tag{27}$$

Then by multiplication $x^2y^2z^2(x+y+z)^3 = t^2p^2q^2$, so

$$x+y+z = \left[\frac{tpq}{xyz(x+y+z)}\right]^2 = v^2,$$

where $tpq = vxyz(x+y+z) = v^3xyz$.

This gives $\frac{xyz}{v^2} = u^2$ so $xyz = u^2v^2$ and $tpq = u^2v^5$. Now $xyv^2 = t^2$, $xzv^2 = p^2$, $yzv^2 = q^2$ give $xy = \left(\frac{t}{v}\right)^2$, where $t = vn_1$, $xz = \left(\frac{p}{v}\right)^2$, where $p = vn_2$, $yz = \left(\frac{q}{v}\right)^2$, where $q = vn_3$ (n_1, n_2, n_3 integers). By $v^3n_1n_2n_3 = u^2v^5$ we get $n_1n_2n_3 = u^2v^2$. By $xy = n_1^2$, $xz = n_2^2$, $yz = n_3^2$, $xyz = u^2v^2$, $x+y+z = v^2$, we get $x = d_1X^2$, $y = d_1Y^2$ (with $(X,Y) = 1$), $n_1 = d_1XY$; $x = d_2U^2$, $z = d_2V^2$, $n_2 = d_2UV$, $(U,V) = 1$; $y = d_3W^2$, $z = d_3\Omega^2$, where $(W,\Omega) = 1$, $n_3 = d_3W\Omega$. From $xyz = d_1^2X^2Y^2d_2V^2 = u^2v^2$ it follows that d_2 is a perfect square. So x is a square, implying that d_1 is a square, implying $y =$ perfect square. Thus $n_3 =$ square, giving $z =$ perfect square. All in all, x, y, z are all perfect squares. Let $x = \alpha^2$, $y = \beta^2$, $z = \gamma^2$. Then

$$\alpha^2 + \beta^2 + \gamma^2 = v^2. \tag{28}$$

From $p - a = \frac{b+c-a}{2} = \alpha^2$, $p - b = y = \frac{a+c-b}{2} = \beta^2$, $p - c = z = \frac{a+b-c}{2} = \gamma^2$ we can easily deduce

$$a = \beta^2 + \gamma^2, \quad b = \alpha^2 + \gamma^2, \quad c = \alpha^2 + \beta^2. \tag{29}$$

Now, the primitive solutions of (28) (i.e. those with $(\alpha, \beta, \gamma) = 1$) are given by (see e.g. [1])

$$\alpha = mk - ns, \quad \beta = ms + nk, \quad \gamma = m^2 + n^2 - k^2 - s^2,$$

$$v = m^2 + n^2 + k^2 + s^2 \tag{30}$$

where m, k, n, s ($mk > ns$, $m^2 + n^2 > k^2 + s^2$) are integers. By supposing $(\alpha, \beta, \gamma) = d$, clearly $\alpha = d\alpha_1$, $b = d\beta_1$, $e = d\gamma_1$ and $d^2(\alpha_1^2 + \beta_1^2 + \gamma_1^2) = v^2$ implies $d^2|v^2$, so $d|v$. Let $v = dv_1$, giving $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = v_1^2$. Thus the general solutions of (28) can be obtained from (30), by multiplying each term of (30) by a common factor d.

These give all H-triangles with the required conditions.

Remarks 2. Many generalized or extensions of Heron triangles or arithmetic problems in geometry were included in paper [6]. The part IV of this series (in preparation) will contain other generalized arithmetic problems in plane or space (e.g. "Heron trapeziums").

References

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Note added in proof. After completing this paper, we learned that **Problem CMJ 354** (College Math. J. **18**(1987), 248) by **Alvin Tirman** asks for the determination of Pythagorean triangles with the property that the triangle formed by the altitude and median corresponding to the hypothenuse is also Pythagorean. It is immediate that the solution of this problem follows from paragraph 1. of our paper.