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SHORT REMARK ON NUMBER THEORY. II

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In [1] they are formulated and proved the following problems:

Problem 1: If a, b, c, d are positive real numbers such that $[n.a] + [n.b] = [n.c] + [n.d]$ for all positive integers n, then $a + b = c + d$.

Problem 2: If a, b, c, d are positive irrational numbers such that $a + b = c + d$ then $[n.a] + [n.b] = [n.c] + [n.d]$ for all positive integers n.

Here we shall generalize these problems.

Problem 1': Let $m \geq 1$ be a natural number, $a_1, a_2, ..., a_m, b_1, b_2, ..., b_m$ be positive real numbers. If $\sum_{i=1}^{m} a_i$ and $\sum_{i=1}^{m} b_i$ are natural numbers and if for some natural number $n > m$

$$\sum_{i=1}^{m} [n.a_i] = \sum_{i=1}^{m} [n.b_i],$$

then

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Proof: For every positive real number x we can write $x = [x] + \{x\}$, where $[x]$ is the integer part of x and $\{x\}$ is the fractional part of x. Then

$$|\sum_{i=1}^{m} a_i - \sum_{i=1}^{m} b_i| = |\frac{1}{n}. \sum_{i=1}^{m} (n.a_i - n.b_i)| = |\frac{1}{n}. \sum_{i=1}^{m} ([n.a_i] - [n.b_i] + \{n.a_i\} - \{n.b_i\})|$$

$$= \frac{1}{n}.| \sum_{i=1}^{m} (\{n.a_i\} - \{n.b_i\})| \leq \frac{1}{n}. \sum_{i=1}^{m} |\{n.a_i\} - \{n.b_i\})| < \frac{m}{n} < 1,$$

i.e.,

$$|\sum_{i=1}^{m} a_i - \sum_{i=1}^{m} b_i| < 1.$$

But, by condition $\sum_{i=1}^{m} a_i$ and $\sum_{i=1}^{m} b_i$ are natural numbers. Therefore

$$\sum_{i=1}^{m} a_i - \sum_{i=1}^{m} b_i = 0,$$

i.e.,

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Problem 1": Let $m \geq 1$ be a natural number, $a_1, a_2, ..., a_m, b_1, b_2, ..., b_m$ be positive real numbers. If for every natural number n

$$\sum_{i=1}^{m} [n.a_i] = \sum_{i=1}^{m} [n.b_i],$$

then

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Proof: As above we obtain that

$$|\sum_{i=1}^{m} a_i - \sum_{i=1}^{m} b_i| < \frac{m}{n}$$

for every natural number n. Let n divergents to ∞ with natural values. Therefore

$$|\sum_{i=1}^{m} a_i - \sum_{i=1}^{m} b_i| \leq 0,$$

from where we obtain that

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i.$$

Obviously, Problem 1 is a partial case of Problem 1' and Problem 1".

Problem 2': Let $m \geq 1$ be a natural number, $a_1, a_2, ..., a_m, b_1, b_2, ..., b_m$ be positive real numbers. If

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i,$$

then for every natural number n

$$|\sum_{i=1}^{m} [n.a_i] - \sum_{i=1}^{m} [n.b_i]| \leq m - 1.$$

Proof: Now, we obtain

$$|\sum_{i=1}^{m} [n.a_i] - \sum_{i=1}^{m} [n.b_i]| = |\sum_{i=1}^{m} (n.a_i - \{n.a_i\}) - \sum_{i=1}^{m} (n.b_i - \{n.b_i\})|$$

$$= |\sum_{i=1}^{m} (\{n.a_i\}) - \{n.b_i\})| \leq \sum_{i=1}^{m} |\{n.a_i\}) - \{n.b_i\}| < m.$$

But $\sum_{i=1}^{m} [n.a_i]$ and $\sum_{i=1}^{m} [n.b_i]$ are natural numbers, from where it follows that

$$|\sum_{i=1}^{m} [n.a_i] - \sum_{i=1}^{m} [n.b_i]| \leq m - 1.$$

When $m = 1$ we obtain Problem 2 in a more genaral form, because the condition for irationalness of a, b, c, d.

REFERENCE:

[1] Honsberger R., Mathematical Gems. III. Mathematical Assoc. of America, 1985.