NNTDM 5 (1999) 3, 94-96

ON THE 100-th, THE 101-st AND THE 102-nd SMARANDACHE'S PROBLEMS

Krassimir T. Atanassov
CLBME - Bulg. Academy of Sci., and MRL, P.O.Box 12, Sofia-1113, Bulgaria
e-mail: krat@bgcict.acad.bg

The 100-th problem from [1] (see also 80-th problem from [2]) is the following:

Square roots:

$$0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5,$$
$$5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 7, 7, \ldots$$

($s_q(n)$ is the superior integer part of square root of n.)
Remark: this sequence is the natural sequence, where each number is repeated $2n+1$ times, because between n^2 (included) and $(n+1)^2$ (excluded) there are $(n+1)^2 - n^2$ different numbers.
Study this sequence.

The 101-st problem from [1] (see also 81-st problem from [2]) is the following:

Cubical roots:

$$0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3,$$
$$3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3,$$
$$3, 3, 4, 4, 4, \ldots$$

($c_q(n)$ is the superior integer part of cubical root of n.)
Remark: this sequence is the natural sequence, where each number is repeated $3n^2 + 3n + 1$ times, because between n^3 (included) and $(n+1)^3$ (excluded) there are $(n+1)^3 - n^3$ different numbers.
Study this sequence.

The 102-nd problem from [1] (see also 82-nd problem from [2]) is the following:

$m-$power roots:
($m_q(n)$ is the superior integer part of $m-$power root of n.)
Remark: this sequence is the natural sequence, where each number is repeated $(n+1)^m - n^m$ times.
Study this sequence.

Below we shall use the usual notation: $[x]$ for the integer part of the real number x.

The author thinks that these are some of the most trivial Smarandache's problems. The n-th term of each of the above sequences is, respectively

$$x_n = [\sqrt{n}],$$

of the second -

$$y_n = [\sqrt[3]{n}],$$

and of the third -

$$z_n = [\sqrt[m]{n}].$$

The checks of these equalities is straightforward, or by induction.
We can easily prove the validity of the following equalities:

$$\sum_{k=1}^{n} (2k+1).k = \frac{n(n+1)(4n+5)}{6}, \tag{1}$$

$$\sum_{k=1}^{n} (3k^2+3k+1).k = \frac{n(n+1)(3n^2+7n+4)}{6}. \tag{2}$$

Now using (1) and (2), we shall show the values of the n-th partial sums

$$X_n = \sum_{k=1}^{n} x_k,$$

$$Y_n = \sum_{k=1}^{n} y_k$$

and

$$Z_n = \sum_{k=1}^{n} z_k,$$

of the three Smarandache's sequences. They are, respectively,

$$X_n = \frac{([\sqrt{n}]-1)[\sqrt{n}](4[\sqrt{n}]+1)}{6} + n - [\sqrt{n}]^2 + 1).[\sqrt{n}], \tag{3}$$

$$Y_n = \frac{([\sqrt[3]{n}-1)[\sqrt[3]{n}]^2(3[\sqrt[3]{n}+1)}{4} + (n - [\sqrt[3]{n}]^3 + 1).[\sqrt[3]{n}], \tag{4}$$

$$Z_n = \sum_{k=1}^{n} (([\sqrt[m]{k}]+1)^m - [\sqrt[m]{k}]^m)[\sqrt[m]{k}-1]^m$$
$$+(n - [\sqrt[m]{n}]^m + 1).[\sqrt[m]{n}]. \tag{5}$$

The proofs can be made by induction. For example, the validity of (3) is proved as follows.
Let n = 1. Then the validity of (3) is obvious. Let us assume that (3) is valid for some natural number n. For the form of n there are two cases:
(a) $n+1$ is not a square. Therefore,

$$[\sqrt{n+1}] = [\sqrt{n}]$$

and then

$$X_{n+1} = X_n + x_{n+1}$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]-1)(4[\sqrt{n}]+1)}{6} + (n-[\sqrt{n}]^2+1).[\sqrt{n}] + [\sqrt{n+1}]$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6} + (n+1-[\sqrt{n+1}]^2+1)$$

$$.[\sqrt{n+1}].$$

(b) $n+1$ is a square (for $n \geq 1$ it follows that n is not a square). Therefore,

$$[\sqrt{n+1}] = [\sqrt{n}] + 1$$

and then

$$X_{n+1} = X_n + x_{n+1}$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]-1)(4[\sqrt{n}]+1)}{6} + (n-[\sqrt{n}]^2+1).[\sqrt{n}] + [\sqrt{n+1}]$$

$$= \frac{([\sqrt{n+1}]-1)([\sqrt{n+1}]-2)(4[\sqrt{n+1}]-3)}{6}$$

$$+(n+1-([\sqrt{n+1}]-1)^2).([\sqrt{n}]-1) + [\sqrt{n+1}]$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6}$$

$$-([\sqrt{n+1}]-1)(2[\sqrt{n+1}]-1)$$

$$+(n+1-([\sqrt{n+1}]-1)^2).([\sqrt{n}]-1) + [\sqrt{n+1}]$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6} + [\sqrt{n+1}]$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6}$$

$$+((n+1)-[\sqrt{n+1}]^2+1).[\sqrt{n+1}].$$

Therefore, (3) is valid.

The validity of formulas (4) and (5) are proved analogically.

REFERENCE:

[1] C. Dumitrescu, V. Seleacu, Some notions and questions in number theory, Erhus Univ. Press, Glendale, 1994.

[2] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.