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ON THE 126-th SMARANDACHE'S PROBLEM

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The following Smarandache's problem is formulated in [1] with the title "Smarandache divisibility theorem":

If a and m are integers, and $m > 0$, then:

$$(a^m - a)(m-1)!$$

is divisible by m.

The proof of this assertion follows directly from the Fermat's (small) theorem (see, e.g. [2]).

Really, let a and m are integers and let $m > 0$.

For m there are two cases:

(a) m is a prime number. Then from the Fermat's theorem follows that $a^m - a$ is divisible by m and therefore

$$A \equiv (a^m - a)(m-1)!$$

is divisible by m.

(b) m is not a prime number. Then $m = p.r$ for the natural numbers r and the prime number p. If $r \neq p$ (r can be as a prime number, as well as a composite number), then $2 \leq p, r \leq m-1$ and $p, r \in \{1, 2, ..., m\}$. Therefore p and r are different divisors of $(m-1)!$ and hence $(m-1)!$ is divisible by m. Hence A is divisible by m, too.

The last case is $r = p$, i.e., r is a prime number and $m = p^2$. Therefore, $(m-1)! = p.B$ for some natural number B and we must prove that

$$a^{p^2} - a = p.C,$$

for some natural number C.

Indeed, if $p|a$, then $p|(a^{p^2} - a)$, i.e.

$$a^{p^2} - a = p.C,$$

for some natural number C. On the other hand, if it is not valid that $p|a$, $p|(a^{p^2} - a)$ too, because of the representation

$$a^{p^2} - a = a.(D^{p-1} - 1),$$

where

$$D = p^{p+1}$$

and the fact that $p|(D^{p-1}-1)$ according to Fermat's small theorem, i.e., again

$$a^{p^2} - a = p.C,$$

for some natural number C.

Therefore,

$$A = p^2.B.C,$$

i.e., A is divisible by m.

Therefore, the "Smarandache divisibility theorem" is valid.

There are other ways for proving the last part of the proof. For example, Dr. Mladen Vassilev - Missana gave the following.

Let $m = p^2$. We remind the Legendre's formula

$$ord_p x! = [\frac{x}{p}] + [\frac{x}{p^2}] + [\frac{x}{p^3}] + ...$$

For $x = m - 1$ we obtain

$$ord_p(m-1)! = ord_p(p^2-1)! = [p - \frac{1}{p}] + [1 - \frac{1}{p^2}] = p - 1.$$

Therefore, $p^2|(m-1)!$ iff $p - 1 \geq 2$, i.e., iff $p \geq 3$.

Then remains only the case $m = 2^2 = 4$. In this case $2 = p|(m-1)!$ and obviously, we have

$$2 = p|(a^m - a) = a^4 - a = a(a^2-1)(a^2+1) = a(a-1)(a+1)(a^2+1),$$

so again it is fulfilled

$$p^2 = m|(a^m-1)(m-1)!.$$

Therefore the problem is solved.

REFERENCES:

[1] Dumitrescu C., Seleacu V., Problem 118. Some Notions and Questions in Number Theory. Erhus Univ. Press, Glendale, 1994.

[2] Nagell T., Introduction to Number Theory. John Wiley & Sons, Inc., New York, 1950.