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A remark of the $h_1(\tau)$-function

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§1. Introduction and preliminaries

Let $\mathbb{C}$ be a complex field, $\tau \in \mathbb{C}$. Let $q = e^{2\pi i\tau}$. Then

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}.([3])$$

Let $\rho := e^{\frac{2\pi i}{3}}$ and $-\overline{\rho} := e^{\frac{2\pi i}{6}}$. Then

$$\Delta(\tau) = 16\pi^{12}\rho \frac{\{\eta(\frac{\tau+1}{2})\eta(\frac{\tau+2}{2})\}^{16}}{\eta(\tau)^{24}} \left(\overline{\rho}\eta(\frac{\tau+1}{2})^8 + \eta(\frac{\tau+2}{2})^8\right)^2 .([2])$$

Also we let

$$h_r(\tau)^{2n} := \frac{1}{(256)^n}\left[2\Omega(\frac{\tau}{2})^3\Omega(\frac{\tau+1}{2})^3 + \Omega(\frac{\tau}{2})^2\Omega(\frac{\tau+1}{2})^2 \cdot \left(\rho\Omega(\frac{\tau+1}{2})^r + \overline{\rho}\Omega(\frac{\tau}{2})^r\right)\right]^n,$$

for arbitrary n and r.

In [1], we considered properties of $h_r(\tau)$-function. We think that the $h_r(\tau)^n$-function is closely connected with a study of θ_3-series and Ramanujan numbers of $\Delta(\tau)$. Let we write $\widetilde{h_1(\tau)}^2 = 256h_1(\tau)^2$.

In this paper, we mainly deal with the following: For each $t \in \mathbb{Z}^+$, we will check that there is n such that t is the coefficient of a term in $\widetilde{h_1(\tau)}^{2n}$ and check how many such n exists. We describe propositions on the $h_1(\tau)$-function which are needed in main result.

Proposition 1. *([1]) Let $\Delta(\tau) = \{\epsilon\Omega(\tau)\}^3$, where $\epsilon = 1, \rho$ or $\overline{\rho}$.*

(a) $$\eta(\tau)^{48} = \frac{1}{256}\left[2\eta(\tfrac{\tau}{2})^{24}\eta(\tfrac{\tau+1}{2})^{24} + \eta(\tfrac{\tau}{2})^{16}\eta(\tfrac{\tau+1}{2})^{16}\left(\rho\eta(\tfrac{\tau+1}{2})^{16} + \overline{\rho}\eta(\tfrac{\tau}{2})^{16}\right)\right].$$

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(b) $\Omega(\tau)^6 = \frac{1}{256}\Omega(\frac{\tau}{2})^2\Omega(\frac{\tau+1}{2})^2\left[\Omega(\frac{\tau+1}{2})\left(\Omega(\frac{\tau}{2})+\rho\Omega(\frac{\tau+1}{2})\right)\right.$

$\left.+\Omega(\frac{\tau}{2})\left(\Omega(\frac{\tau+1}{2})+\bar{\rho}\Omega(\frac{\tau}{2})\right)\right].$

Proposition 2. *([1]) For all integers $n \geq 1$,*

$$h_1(\tau)^{2n} = \frac{1}{(256)^n}\left(P + \sum_{i=0}^{[\frac{n}{3}]-1} T_i\left(\Omega(\frac{\tau+1}{2})^{3(i+1)} + \Omega(\frac{\tau}{2})^{3(i+1)}\right)\right.$$

$$+ \sum_{j=0}^{[\frac{\bar{\epsilon}(n)-1}{3}]} Q_j\left(\rho\Omega(\frac{\tau+1}{2})^{3j+1} + \bar{\rho}\Omega(\frac{\tau}{2})^{3j+1}\right)$$

$$\left.+ \sum_{k=0}^{[\frac{\tilde{\epsilon}(n)-2}{3}]} R_k\left(\rho\Omega(\frac{\tau}{2})^{3k+2} + \bar{\rho}\Omega(\frac{\tau+1}{2})^{3k+2}\right)\right),$$

where P, T_i, Q_j and $R_k \in \mathbb{Z}[\Omega(\frac{\tau}{2})^a\Omega(\frac{\tau+1}{2})^a]$, $a, i, j, k \in \mathbb{Z}$, and [ ] is the greatest integer function, where

$$\bar{\epsilon}(n) = \begin{cases} 3t+1, & \text{if } n = 3t+1 \text{ or } 2, \\ 3(t-1)+1, & \text{if } n = 3t, \end{cases}$$

and $$\tilde{\epsilon}(n) = \begin{cases} 3t+2, & \text{if } n = 3t+2, \\ 3(t-1)+2, & \text{if } n = 3t \text{ or } +1. \end{cases}$$

In particular,

$$P = \sum_{t=0}^{\frac{\epsilon'(n)}{2}} 2^{n-2t}\binom{n}{2t}\binom{2t}{t}\Omega(\frac{\tau}{2})^{3n-t}\Omega(\frac{\tau+1}{2})^{3n-t},$$

where $$\epsilon'(n) = \begin{cases} n, & \text{if } n \text{ is even}, \\ n-1, & \text{if } n \text{ is odd} \end{cases}$$

and the binomial coefficient $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$.

§2. Main result

For each $t \in \mathbb{Z}^+$, we will check that there is n such that t is the coefficient of a term in $\widetilde{h_1(\tau)}^{2n}$ and check how many such n exists.

Theorem.

(1) *For each $t \in \mathbb{Z}^+$, there is n such that t is the coefficient of a term in $\widetilde{h_1(\tau)}^{2n}$.*

(2) *$t = 1$ is the coefficient of a term in $\widetilde{h_1(\tau)}^{2n}$ for all n.*

In particular, 1 is the coefficient of

$$\Omega(\frac{\tau+1}{2})^{2n}\Omega(\frac{\tau}{2})^{2n}(\bar{\epsilon}\Omega(\frac{\tau+1}{2})^n + \bar{\epsilon}'\Omega(\frac{\tau}{2})^n) \text{ in } \widetilde{h_1(\tau)}^{2n},$$

$$\text{where} \begin{cases} \hat{\epsilon} = \hat{\epsilon}' = 1 & \text{if } n \equiv 0(3), \\ \hat{\epsilon} = \rho, \hat{\epsilon}' = \bar{\rho} & \text{if } n \equiv 1(3), \\ \hat{\epsilon} = \bar{\rho}, \hat{\epsilon}' = \rho & \text{if } n \equiv 2(3). \end{cases}$$

(3) *If t is an odd prime, then there is unique integer n such that t is the coefficient of*

$$\Omega(\frac{\tau+1}{2})^{2n+1}\Omega(\frac{\tau}{2})^{2n+1}(\hat{\epsilon}\Omega(\frac{\tau+1}{2})^{n-2} + \hat{\epsilon}'\Omega(\frac{\tau}{2})^{n-2}) \text{ in } \widetilde{h_1(\tau)}^{2n}.$$

In particular,

$$\begin{cases} \text{if } t = 2 & \text{then } n = 1, \ \Omega(\frac{\tau+1}{2})^3\Omega(\frac{\tau}{2})^3 \\ & \text{and } n = 2, \ \Omega(\frac{\tau+1}{2})^5\Omega(\frac{\tau}{2})^5, \\ \text{if } t = 3 & \text{then } n = 3, \ \Omega(\frac{\tau+1}{2})^7\Omega(\frac{\tau}{2})^7(\rho\Omega(\frac{\tau+1}{2}) + \bar{\rho}\Omega(\frac{\tau}{2})), \\ \text{if } t \equiv 1 \ (3) & \text{then } n = t, \ \Omega(\frac{\tau+1}{2})^{2t+1}\Omega(\frac{\tau}{2})^{2t+1}(\bar{\rho}\Omega(\frac{\tau+1}{2})^{t-2} + \rho\Omega(\frac{\tau}{2})^{t-2}), \\ \text{if } t \equiv 2 \ (3) & \text{then } n = t, \ \Omega(\frac{\tau+1}{2})^{2t+1}\Omega(\frac{\tau}{2})^{2t+1}(\Omega(\frac{\tau+1}{2})^{t-2} + \Omega(\frac{\tau}{2})^{t-2}). \end{cases}$$

Proof.

(1) Let t be a positive integer. All coefficients of $\widetilde{h_1(\tau)}^{2t}$ are forms of

$$\binom{n}{i}2^{n-i}\binom{i}{j},$$

for some i, j and $n \in \mathbb{Z}$. For each integer,

$$t = \binom{t}{t}2^{t-t}\binom{t}{t-1}.$$

Therefore t is the coefficient of $\widetilde{h_1(\tau)}^{2t}$.

(2) Let $t = 1$ and $n \in \mathbb{Z}^+$. Then since

$$1 = \binom{n}{n}2^{n-n}\binom{n}{n},$$

1 is the coefficients of

$$\Omega(\frac{\tau+1}{2})^{2n}\Omega(\frac{\tau}{2})^{2n}((\rho\Omega(\frac{\tau+1}{2}))^n + (\bar{\rho}\Omega(\frac{\tau}{2}))^n) \text{ in } \widetilde{h_1(\tau)}^{2n}.$$

Hence, this completes (2).

(3) Let $t = 2$, i.e.,

$$2 = \binom{n}{i}2^{n-i}\binom{i}{j}, \text{ for some } i, j \text{ and } n.$$

Since 2 and 1 are factors of 2, which of both satisfy

$$\binom{n}{i} = 1,\ 2^{n-i} = 2,\ \binom{i}{j} = 1 \text{ or } \binom{n}{i} = 1,\ 2^{n-i} = 1,\ \binom{i}{j} = 2.$$

I.e., we have $n = 1,\ i = 0 = j$ or $n = i = 2,\ j = 1$.

Therefore, 2 is the coefficient of $\widetilde{h_1(\tau)}$ and $\widetilde{h_1(\tau)}^2$ in $\Omega(\frac{\tau+1}{2})^3\Omega(\frac{\tau}{2})^3$ and $\Omega(\frac{\tau+1}{2})^5\Omega(\frac{\tau}{2})^5$, respectively.

Let t be an odd prime. Then

$$t = \binom{n}{i}2^{n-i}\binom{i}{j} \text{ for some } i, j \text{ and } n.$$

Since t is an odd prime, $n = i$, $\binom{n}{i} = 1$ and $\binom{i}{j} = t$.

Hence $i = t$ and $j = 1$.

By Proposition 2, we will easily check the remainder of (3). □

We write examples of Theorem in Appendix A.

Corollary. *If $t \equiv 1\ (3)$ is a prime then there is no t such that t is the coefficient of $\Omega(\frac{\tau+1}{2})^{2t+1}\Omega(\frac{\tau}{2})^{2t+1}(\rho\Omega(\frac{\tau+1}{2})^{t-2} + \overline{\rho}\Omega(\frac{\tau}{2})^{t-2})$ in $\widetilde{h_1(\tau)}^{2n}$.*

Proof. By Theorem, it is trivial. □

Also, if t is an even integer, then there exists at least two n's such that t is the coefficient of a term in $\widetilde{h_1(\tau)}^{2n}$.

For $n = i = t$, $j = 1$ and $n = \frac{t}{2}$, $j = i = \frac{t}{2} - 1$.

If t is an odd prime, then there is only integer n such that t^2 is the coefficient of a term in $\widetilde{h_1(\tau)}^{2n}$.

Appendix A. Examples of Theorem

t	n	term
1	1	$\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2(\rho\Omega(\frac{\tau+1}{2}) + \overline{\rho}\Omega(\frac{\tau}{2}))$
	2	$\Omega(\frac{\tau+1}{2})^4\Omega(\frac{\tau}{2})^4(\overline{\rho}\Omega(\frac{\tau+1}{2})^2 + \rho\Omega(\frac{\tau}{2})^2)$
	3	$\Omega(\frac{\tau+1}{2})^6\Omega(\frac{\tau}{2})^6(\Omega(\frac{\tau+1}{2})^3 + \Omega(\frac{\tau}{2})^3)$
	⋮	⋮
	b	$\Omega(\frac{\tau+1}{2})^{2b}\Omega(\frac{\tau}{2})^{2b}((\rho\Omega(\frac{\tau+1}{2}))^b + (\overline{\rho}\Omega(\frac{\tau}{2}))^b)$
	⋮	⋮
2	1	$\Omega(\frac{\tau+1}{2})^3\Omega(\frac{\tau}{2})^3$
	2	$\Omega(\frac{\tau+1}{2})^5\Omega(\frac{\tau}{2})^5$
3	3	$\Omega(\frac{\tau+1}{2})^7\Omega(\frac{\tau}{2})^7(\rho\Omega(\frac{\tau+1}{2}) + \overline{\rho}\Omega(\frac{\tau}{2}))$
4	2	$\Omega(\frac{\tau+1}{2})^5\Omega(\frac{\tau}{2})^5(\rho\Omega(\frac{\tau+1}{2}) + \overline{\rho}\Omega(\frac{\tau}{2}))$
	2	$\Omega(\frac{\tau+1}{2})^6\Omega(\frac{\tau}{2})^6$

	4	$\Omega(\frac{\tau+1}{2})^9\Omega(\frac{\tau}{2})^9(\overline{\rho}\Omega(\frac{\tau+1}{2})^2+\rho\Omega(\frac{\tau}{2})^2)$
5	5	$\Omega(\frac{\tau+1}{2})^{11}\Omega(\frac{\tau}{2})^{11}(\Omega(\frac{\tau+1}{2})^3+\Omega(\frac{\tau}{2})^3)$
6	3	$\Omega(\frac{\tau+1}{2})^7\Omega(\frac{\tau}{2})^7(\overline{\rho}\Omega(\frac{\tau+1}{2})^2+\rho\Omega(\frac{\tau}{2})^2)$
	4	$\Omega(\frac{\tau+1}{2})^{10}\Omega(\frac{\tau}{2})^{10}$
	6	$\Omega(\frac{\tau+1}{2})^{13}\Omega(\frac{\tau}{2})^{13}(\rho\Omega(\frac{\tau+1}{2})^4+\overline{\rho}\Omega(\frac{\tau}{2})^4)$
7	7	$\Omega(\frac{\tau+1}{2})^{15}\Omega(\frac{\tau}{2})^{15}(\overline{\rho}\Omega(\frac{\tau+1}{2})^5+\rho\Omega(\frac{\tau}{2})^5)$
8	3	$\Omega(\frac{\tau+1}{2})^9\Omega(\frac{\tau}{2})^9$
	4	$\Omega(\frac{\tau+1}{2})^9\Omega(\frac{\tau}{2})^9(\Omega(\frac{\tau+1}{2})^3+\Omega(\frac{\tau}{2})^3)$
	8	$\Omega(\frac{\tau+1}{2})^{17}\Omega(\frac{\tau}{2})^{17}(\Omega(\frac{\tau+1}{2})^6+\Omega(\frac{\tau}{2})^6)$
9	9	$\Omega(\frac{\tau+1}{2})^{19}\Omega(\frac{\tau}{2})^{19}(\rho\Omega(\frac{\tau+1}{2})^7+\overline{\rho}\Omega(\frac{\tau}{2})^7)$
10	5	$\Omega(\frac{\tau+1}{2})^{11}\Omega(\frac{\tau}{2})^{11}(\rho\Omega(\frac{\tau+1}{2})^4+\overline{\rho}\Omega(\frac{\tau}{2})^4)$
	5	$\Omega(\frac{\tau+1}{2})^{12}\Omega(\frac{\tau}{2})^{12}(\rho\Omega(\frac{\tau+1}{2})+\overline{\rho}\Omega(\frac{\tau}{2}))$
	10	$\Omega(\frac{\tau+1}{2})^{21}\Omega(\frac{\tau}{2})^{21}(\overline{\rho}\Omega(\frac{\tau+1}{2})^8+\rho\Omega(\frac{\tau}{2})^8)$
11	11	$\Omega(\frac{\tau+1}{2})^{23}\Omega(\frac{\tau}{2})^{23}(\Omega(\frac{\tau+1}{2})^9+\Omega(\frac{\tau}{2})^9)$
12	3	$\Omega(\frac{\tau+1}{2})^8\Omega(\frac{\tau}{2})^8(\rho\Omega(\frac{\tau+1}{2})+\overline{\rho}\Omega(\frac{\tau}{2}))$
	6	$\Omega(\frac{\tau+1}{2})^{13}\Omega(\frac{\tau}{2})^{13}(\overline{\rho}\Omega(\frac{\tau+1}{2})^5+\rho\Omega(\frac{\tau}{2})^5)$
	12	$\Omega(\frac{\tau+1}{2})^{25}\Omega(\frac{\tau}{2})^{25}(\rho\Omega(\frac{\tau+1}{2})^{12}+\overline{\rho}\Omega(\frac{\tau}{2})^{12})$
13	13	$\Omega(\frac{\tau+1}{2})^{27}\Omega(\frac{\tau}{2})^{27}(\overline{\rho}\Omega(\frac{\tau+1}{2})^{11}+\rho\Omega(\frac{\tau}{2})^{11})$
14	7	$\Omega(\frac{\tau+1}{2})^{15}\Omega(\frac{\tau}{2})^{15}(\Omega(\frac{\tau+1}{2})^6+\Omega(\frac{\tau}{2})^6)$
	14	$\Omega(\frac{\tau+1}{2})^{29}\Omega(\frac{\tau}{2})^{29}(\Omega(\frac{\tau+1}{2})^{12}+\Omega(\frac{\tau}{2})^{12})$
15	6	$\Omega(\frac{\tau+1}{2})^{14}\Omega(\frac{\tau}{2})^{14}(\overline{\rho}\Omega(\frac{\tau+1}{2})^2+\rho\Omega(\frac{\tau}{2})^2)$
	15	$\Omega(\frac{\tau+1}{2})^{31}\Omega(\frac{\tau}{2})^{31}(\rho\Omega(\frac{\tau+1}{2})^{13}+\overline{\rho}\Omega(\frac{\tau}{2})^{13})$
16	4	$\Omega(\frac{\tau+1}{2})^{12}\Omega(\frac{\tau}{2})^{12}$
	8	$\Omega(\frac{\tau+1}{2})^{17}\Omega(\frac{\tau}{2})^{17}(\rho\Omega(\frac{\tau+1}{2})^7+\overline{\rho}\Omega(\frac{\tau}{2})^7)$
	16	$\Omega(\frac{\tau+1}{2})^{33}\Omega(\frac{\tau}{2})^{33}(\overline{\rho}\Omega(\frac{\tau+1}{2})^{14}+\rho\Omega(\frac{\tau}{2})^{14})$
17	17	$\Omega(\frac{\tau+1}{2})^{35}\Omega(\frac{\tau}{2})^{35}(\Omega(\frac{\tau+1}{2})^{15}+\Omega(\frac{\tau}{2})^{15})$
18	9	$\Omega(\frac{\tau+1}{2})^{19}\Omega(\frac{\tau}{2})^{19}(\overline{\rho}\Omega(\frac{\tau+1}{2})^8+\rho\Omega(\frac{\tau}{2})^8)$
	18	$\Omega(\frac{\tau+1}{2})^{37}\Omega(\frac{\tau}{2})^{37}(\rho\Omega(\frac{\tau+1}{2})^{16}+\overline{\rho}\Omega(\frac{\tau}{2})^{16})$
19	19	$\Omega(\frac{\tau+1}{2})^{39}\Omega(\frac{\tau}{2})^{39}(\overline{\rho}\Omega(\frac{\tau+1}{2})^{17}+\rho\Omega(\frac{\tau}{2})^{17})$
20	6	$\Omega(\frac{\tau+1}{2})^{15}\Omega(\frac{\tau}{2})^{15}$
	10	$\Omega(\frac{\tau+1}{2})^{21}\Omega(\frac{\tau}{2})^{21}(\Omega(\frac{\tau+1}{2})^9+\Omega(\frac{\tau}{2})^9)$
	20	$\Omega(\frac{\tau+1}{2})^{41}\Omega(\frac{\tau}{2})^{41}(\Omega(\frac{\tau+1}{2})^{18}+\Omega(\frac{\tau}{2})^{18})$

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