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ONE EXTREMAL PROBLEM. 8

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The following problem is similar to the problems from [1-4], but it is not less interesting than the last ones: to determine the value for K for which $m_K = \max\limits_{1 \le k \le n} m_k$, where $n \ge 1$ is a fixed natural number $m_k = (\frac{n}{k})^k$.

Directly it is checked that $m_1 = n$, $m_2 = n^2/4$, $m_{n-1} = (\frac{n}{n-1})^{n-1}$, $m_n = 1$. Therefore, there exists at least one k $(1 \le k \le n)$ for which:

$$\begin{aligned} m_{k-1} &< m_k \\ m_{k+1} &\le m_k \end{aligned} \qquad (1)$$

Below we shall use the inequalities (see [5]):

$$e.\frac{2.x}{2.x+1} < (1 + \frac{1}{x})^x < e.\frac{2.x+1}{2.x+2} \qquad (2)$$

Let us assume the existence of a natural number q $(1 \le q \le N)$ for which:

$$\begin{aligned} m_{q-1} &\ge m_q \\ m_{q+1} &\ge m_q \end{aligned} \qquad (3)$$

i.e.,

$$\begin{aligned} (\frac{n}{q-1})^{q-1} &\ge (\frac{n}{q})^q \\ (\frac{n}{q+1})^{q+1} &\ge (\frac{n}{q})^q . \end{aligned}$$

Therefore, the following two inequalities are valid simultaneously (see (2)):

$$\begin{aligned} \frac{n}{q} &\le (\frac{q}{q-1})^{q-1} < e.\frac{2.q-1}{2.q} \\ \frac{n}{q+1} &\ge (\frac{q+1}{q})^q > e.\frac{2.q}{2.q+1} \end{aligned}$$

and hence from the first two inequalities follows that

$$q > \frac{n}{e} + \frac{1}{2} \qquad (4)$$

and from the second two inequalities follows that

$$\frac{n}{e} > \frac{2.q^2 + 2.q}{2.q+1}$$

from where it follows that

$$q < \frac{n}{2.e} - \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} + 1} . \qquad (5)$$

From (4) and (5) it follows that

$$\frac{n}{e} + \frac{1}{2} < \frac{n}{2.e} - \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} + 1}$$

which is not true. Therefore, there are not three numbers m_{q-1}, m_q and m_{q+1} for which (3) is valid. In particular, there are not three numbers for which $m_{q-1} = m_q = m_{q+1}$.

The following question is interesting, too: is there a natural number k $(1 \le k \le N)$ for which

$$m_{k+1} = m_k \ ?$$

Let k have this property. Then

$$(\frac{n}{k+1})^{k+1} = (\frac{n}{k})^k$$

i.e.

$$n^{k+1} . k^k = (k + 1)^{k+1} .$$

But k is not a divisor of k + 1, i.e. the equality can be valid only for k = 1 and n = 2.

Therefore the system (1) has the form

$$m_{k-1} < m_k$$

i.e. (cf. [1]):

$$m_{k+1} < m_k \qquad (6)$$

$$m_0 < m_1 \ldots < m_{k-1} < m_k > m_{k+2} \ldots > m_N ,$$

and from there it is seen that K = k.

By the way above, but using the formula:

$$e.\frac{2.x}{2.x + 1} < (1 + \frac{1}{x})^x < e.\frac{4.x}{4.x + 1} \qquad (7)$$

(see [5]) it can easily be seen that:

$$\frac{n}{k} > (\frac{k}{k - 1})^{k-1} > e.\frac{2.k - 2}{2.k - 1}$$

$$\frac{n}{k + 1} < (\frac{k + 1}{k})^k < e.\frac{4.k}{4.k + 1}$$

Hence from the first two inequalities follows that

$$k < \frac{n}{e}.\frac{2.k - 1}{2.k - 2} ,$$

from where it follows that

$$k < \frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}. \quad \frac{n^2}{e^2} + 1$$

i.e.,

$$k \leq [\frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} + 1}] \qquad (8)$$

and from the second two inequalities follows that

$$k > \frac{n}{2.e} - \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} - \frac{n}{e} + 1}$$

i.e.

$$k \geq [\frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} - \frac{n}{e} + 1}] \qquad (9)$$

Finally, from (1), (8) and (9) it follows that for K the following inequalities are valid:

$$[\frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}. \quad \frac{n^2}{e^2} - \frac{n}{e} + 1] \leq K \leq [\frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} + 1}].$$

and easily can be seen that:

$$(\frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}. \quad \frac{n^2}{e^2} + 1) - (\frac{n}{2.e} + \frac{1}{2} + \frac{1}{2}.\sqrt{\frac{n^2}{e^2} - \frac{n}{e} + 1}) < \frac{1}{2}.$$

Therefore for every natural number n there exists exactly one value of K.

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