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NUMERICAL PROPERTIES OF MORGAN-VOYCE NUMBERS

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1. INTRODUCTION

A general class of number sets $\{X_n\}$ is defined recursively by

$$X_n = 3X_{n-1} - X_{n-2} \tag{1.1}$$

with

$$X_0 = a, \ X_1 = b \ \ (a, b \text{ integers}). \tag{1.2}$$

Particular cases arise as follow:

	X_n	a	b	
(B)	B_n	0	1	
(b)	b_n	1	1	(1.3)
(C)	C_n	2	3	
(c)	c_n	-1	1.	

Cases (B), (b) give Morgan-Voyce numbers (used in ladder network analysis [4]), while (C), (c) produce number sets related to these. All four cases are specializations of corresponding polynomials $B_n(x), b_n(x), C_n(x)$, and $c_n(x)$ [3] when $x = 1$. Most of the results in this article were suppressed from [3] but are presented here as possibly having an interest *per se*.

2. BASICS

From (1.1), the roots of the characteristic equation

$$\lambda^2 - 3\lambda + 1 = 0 \tag{2.1}$$

are clearly

$$\alpha = \frac{3+\sqrt{5}}{2}, \ \beta = \frac{3-\sqrt{5}}{2} \tag{2.2}$$

whence

$$\alpha\beta = 1, \ \alpha + \beta = 3, \ \alpha - \beta = \sqrt{5} = \Delta. \tag{2.3}$$

Binet forms for $B_n, \cdots, c_n$ in (1.3) are

$$B_n = \frac{\alpha^n - \beta^n}{\Delta} \tag{2.4}$$

$$b_n = \frac{(1-\beta)\alpha^n - (1-\alpha)\beta^n}{\Delta} = B_n - B_{n-1} \tag{2.5}$$

$$C_n = \alpha^n + \beta^n \tag{2.6}$$

$$c_n = \frac{(1+\beta)\alpha^n - (1+\alpha)\beta^n}{\Delta} = B_n + B_{n-1}. \tag{2.7}$$

Admitting negative values of n to our definitions (1.1)-(1.3), we deduce from (2.3)-(2.7) that

$$B_{-n} = -B_n \tag{2.8}$$

$$b_{-n} = b_{n+1} \tag{2.9}$$

$$C_{-n} = C_n \tag{2.10}$$

$$c_{-n} = -c_{n+1}. \tag{2.11}$$

3. SOME INTERESTING RELATIONSHIPS

Most of the following results are derivable from the recurrence relations (1.1)-(1.3) and/or the Binet forms (2.4)-(2.7), with (2.3). The notations F_n, L_n stand for the nth Fibonacci and nth Lucas numbers, respectively.

$$B_n = F_{2n} \tag{3.1}$$

$$C_n = L_{2n} \tag{3.2}$$

$$b_n = F_{2n-1} \tag{3.3}$$

$$c_n = L_{2n-1} \tag{3.4}$$

$$B_n C_n = B_{2n} \tag{3.5}$$

$$b_n c_n = B_{2n-1} \tag{3.6}$$

$$B_{n+1} - B_{n-1} = C_n \tag{3.7}$$

$$C_{n+1} - C_{n-1} = 5B_n \tag{3.8}$$

$$b_{n+1} - b_{n-1} = c_n \tag{3.9}$$

$$c_{n+1} - c_{n-1} = 5b_n \tag{3.10}$$

$$b_{n+1} - b_n = B_n \tag{3.11}$$

$$c_{n+1} - c_n = C_n \tag{3.12}$$

$$C_n = 2B_n - 3B_{n-1} \tag{3.13}$$

$$5B_n = 2C_{n+1} - 3C_n \tag{3.14}$$

Simson formulas

$$B_{n+1}B_{n-1} - B_n^2 = -(b_{n+1}b_{n-1} - b_n^2) = -1 \tag{3.15}$$

$$C_{n+1}C_{n-1} - C_n^2 = -(c_{n+1}c_{n-1} - c_n^2) = 5 \tag{3.16}$$

Generating functions

$$\textstyle\sum_{i=1}^{\infty} B_i y^{i-1} = (1 - \overline{3y - y^2})^{-1} \tag{3.17}$$

$$\textstyle\sum_{i=1}^{\infty} b_i y^{i-1} = (1 - 2y)(1 - \overline{3y - y^2})^{-1} \tag{3.18}$$

$$\textstyle\sum_{i=0}^{\infty} C_i y^{i} = (2 - 3y)(1 - \overline{3y - y^2})^{-1} \tag{3.19}$$

$$\textstyle\sum_{i=0}^{\infty} c_i y^{i} = (-1 + 4y)(1 - \overline{3y - y^2})^{-1} \tag{3.20}$$

Closed forms

$$B_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \tag{3.21}$$

$$b_n = \sum_{k=0}^{n-1} \binom{n+k-1}{2k} \tag{3.22}$$

$$C_n = \sum_{k=0}^{n-1} \frac{2n}{n-k} \binom{n+k-1}{n-k-1} + 1 \tag{3.23}$$

$$c_n = \sum_{k=1}^{n} \frac{2n-1}{2k-1} \binom{n+k-2}{n-k} \tag{3.24}$$

Summations

$$\textstyle\sum_{i=1}^{n} B_i = b_{n+1} - 1 = F_{2n+1} - 1 \tag{3.25}$$

$$\textstyle\sum_{i=1}^{n} b_i = B_{2n} = F_{2n} \tag{3.26}$$

$$\textstyle\sum_{i=1}^{n} C_i = c_{n+1} - 1 = L_{2n+1} - 1 \tag{3.27}$$

$$\textstyle\sum_{i=1}^{n} c_i = C_n - 2 = L_{2n} - 2. \tag{3.28}$$

4. SPECIAL NUMERICAL PROPERTIES

Recurrence (1.1) is a particular case, when $x = 1$, of the polynomial recurrence

$$X_n(x) = (2 + x)X_{n-1}(x) - X_{n-2}(x) \tag{4.1}$$

with initial conditions (1.2). Details of this generalization occur in [3].

Specializations of (4.1) when $x \neq 1$ which are of some interest arise when, say $x = -4, x = -3, x = -2, x = -1, x = 0, x = 2, x = 3, x = 4, x = 5, x = 6, x = 8$. A few of these facets of the theory, most of which have already been recorded in [3], are here reproduced for the reader's convenience.

(i) Historical appearances of $\{B_n(6)\}; \frac{1}{2}\{C_n(6)\}, \{b_n(6)\}, \{c_n(6)\}, \{B_n(8)\}$, and $\frac{1}{2}\{C_n(8)\}$ are to be seen in [5]. Seldom are these occurences of more than a century's antiquity.

(ii) $x = 5$ generates Fibonacci and Lucas numbers, e.g. $c_n(5) = F_{4n-2}, b_n(5) = \frac{1}{3}L_{4n-2}$.

(iii) $x = 4$ gives rise to Pell (P_n) and Pell-Lucas (Q_n) numbers, e.g., $b_n(4) = P_{2n-1}, c_n(4) = \frac{1}{2}Q_{2n-1}$.

(iv) $\{B_n(2)\}, \frac{1}{2}\{C_n(2)\}$ appear in [1, p.167].

(v) $\{b_n(2)\}, \{c_n(2)\}$ are listed in Euler [2,p.375].

(vi) $\underline{x=-1}$: $\{B_n(-1)\} \longrightarrow$ sextuple $\{0,1,1,0,-1,-1\}$ repeated *ad. inf.*,
$\{C_n(-1)\} \longrightarrow$ sextuple $\{2,1,-1,-2,-1,1\}$ repeated *ad.inf.*,
$b_n(-1) = B_{n+1}(-1)$,
$c_n(-1) = -C_{n+1}(-1)$.

(vii) $\underline{x=-2:}$ $\{B_n(-2)\} \longrightarrow$ quadruple $\{0,1,0,-1\}$ repeated *ad. inf.*,
$\{b_n(-2)\} \longrightarrow$ quadruple $\{1,-1,-1,1\}$ repeated *ad inf.*,
$C_n(-2) = 2B_{n+1}(-2)$,
$c_n(-2) = -b_{n+1}(-2)$.

(viii) $\underline{x=-3:}$ $\{B_n(-3)\} \longrightarrow$ triple $\{0,1,-1\}$ repeated *ad inf.*,
$\{C_n(-3)\} \longrightarrow$ triple $\{2,-1,-1\}$ repeated *ad.inf.*,
$b_n(-3) = -C_{n+1}(-3)$,
$c_n(-3) = -B_{n+1}(-3)$.

(ix) $\underline{x=-4:}$ $B_{2n+1}(-4) = (-1)^n b_{n+1}(-4) = 2n+1$,
$C_{2n+1}(-4) = 2c_n(-4) = (-1)^{n+1}2$.

Other possible periodicity aspects (as in (vi)–(viii) above) might be investigated.

Divisibility properties of the $X_n(x)$ in (4.1) in conjunction with (1.2), are indicated in [3].

Inevitably, some historical results are resurrected and their features given new life, e.g., in the *American Mathematical Monthly*, Vol.24 (1917), pp.82–3, the problem of finding the general term and sum to n terms of $\{B_n(2)\}$ is posed by George Sosnow (Newark, New Jersey) and solved by William Hoover (Columbus, Ohio).

Lastly, if we allow the symbolism ($k \geq 1$)

$$B_n^{(k)} = B_{n+1}^{(k-1)} - B_{n-1}^{(k-1)} \tag{4.2}$$

and

$$C_n^{(k)} = C_{n+1}^{(k-1)} - C_{n-1}^{(k-1)}, \tag{4.3}$$

in which $B_n^{(0)} = B_n, C_n^{(0)} = C_n$, then, by (3.7) and (3.8), we eventually derive the elegant and compact connections

$$\begin{cases} B_n^{(2k)} &= C_n^{(2k-1)} = 5^k B_n, \\ B_n^{(2k+1)} &= C_n^{(2k)} = 5^k C_n. \end{cases} \tag{4.4}$$

Altogether, the theory flowing from (1.1), (1.2), and (4.1) presented in this paper obviously affords us further avenues for development while providing us with ample scope for simple mathematical pleasures.

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