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THE MODULAR RING $\mathbb{Z}_6$ AND THE AREA OF A PYTHAGOREAN TRIANGLE

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Fermat used the method of infinite descent to show that the area of a Pythagorean triangle can never be a square. Here we use the modular ring $\mathbb{Z}_6$ [1] to obtain the same result.

For a Pythagorean triple $< c, b, a >$, with $c > b > a$,

$$c^2 = b^2 + a^2 \tag{1}$$

The area of the corresponding Pythagorean triangle is $\frac{1}{2}ab$ and the components (within $\mathbb{Z}_6$) fall into one of the equivalence classes $\overline{1}$,$\overline{2}$,$\overline{3}$,$\overline{4}$,$\overline{5}$,$\overline{6}$, each of which follows the has the form [1]

$$6r_i{+}(i-3) \tag{2}$$

where i is the class [1] and r represents the row within $\mathbb{Z}_6$. Even i gives odd integers, odd i gives even integers and there is no $x : x^2 \epsilon \{\overline{2},\overline{5}\}$.

So that, for Pythagorean triangles, $\frac{1}{2}ab = \overline{6}$, which if a square is represented by $(6r'_6 + 3)^2$.

Equation ((1) is satisfied [1] by the 8 sets of primitive Pythagorean triple $< c, b, a >$ with $ab = ba$:

$$< \overline{2}\,\overline{1}\,\overline{6} > , \; < \overline{2}, \overline{5}, \overline{6} > , \; < \overline{4}\,\overline{3}\,\overline{4} > , \; < \overline{4}\,\overline{3}\,\overline{2} >$$

when b is even, and

$$< \overline{2}\;\overline{6}\;\overline{1} > , \; < \overline{2}\;\overline{6}\;\overline{5} > , \; < \overline{4}\;\overline{4}\;\overline{3} > , \; < \overline{4}\,\overline{2}\,\overline{3} >$$

when b is odd $ab{=}ba$.

Substituting equation (2) into equation (1) for the sets $<\bar{2}\bar{1}\bar{6}>$ $<\bar{2}\bar{6}\bar{1}>$ and $<\bar{2}\bar{5}\bar{6}>$, $<\bar{2}\bar{6}\bar{5}>$ gives the parities of r_1 and r_5 shown in Table 1. However, these parities are inconsistent with $\frac{1}{2}ab$ equated to $(6r'_6+3)^2$ (Table 2) so that no integer solution is possible for these sets if the area is taken as a square.

For the sets $<\bar{4}\ \bar{3}\ \bar{4}>$, $<\bar{4}\ \bar{4}\ \bar{3}>$, and $<\bar{4}\ \bar{3}\ \bar{2}>$, $<\bar{4}\ \bar{2}\ \bar{3}>$ the parity of r_3 is obtained from the z-j grid [2] where $z = c - b$; $z(\text{odd}) = (2t-1)^2$, $z(\text{even}) = 2t^2$, $t \epsilon \mathbb{Z}_+$.

The component in class $\bar{3}$ is given by

$2j(j + z^{\frac{1}{2}})$ when z is odd, and

$2(z/2)^{\frac{1}{2}}\ (\ (z/2)^{\frac{1}{2}} + (2j-1)\ )$ when z is even, $j = 1, 2, 3 \ldots.$

so that the parity of r_3 is always even (Table 1). However, if $\frac{1}{2}ab$ is a square in class $\bar{6}$, r_3 would be odd (Table 2) so that the area of the Pythagorean triangle cannot be a square for these sets either.

CLASS SETS for c, b, a	FUNCTIONS	PARITIES of r_i
$\bar{2}\ \bar{1}\ \bar{6}$ $\bar{2}\ \bar{6}\ \bar{1}$	$r_2(3r_2-1) = 3r_6(r_6+1) - 2r_1 + 3{r_1}^2+1)$	r_1 odd
$\bar{2}\ \bar{5}\ \bar{6}$ $\bar{2}\ \bar{6}\ \bar{5}$	$r_2(3r_2-1) = 3r_6(r_6+1) + 2r_5 + 3{r_5}^2+1)$	r_5 odd
$\bar{4}\ \bar{3}\ \bar{4}$ $\bar{4}\ \bar{3}\ \bar{2}$ z odd	$r_3 = j(j + z^{\frac{1}{2}}/3$	r_3 even
$\bar{4}\ \bar{4}\ \bar{3}$ $\bar{4}\ \bar{2}\ \bar{3}$ z even	$r_3 = (z/2)^{\frac{1}{2}}(\,(z/2)^{\frac{1}{2}} + (2j-1)\,)/3$	r_3 even

Table 1

CLASS SETS for c, b, a	$\frac{1}{2}ab$	PARITIES OF r_1
$\overline{2}$ $\overline{1}$ $\overline{6}$ $\overline{2}$ $\overline{6}$ $\overline{1}$	$\frac{1}{2}(6r_1 - 2)(6r_6 + 3)$	residual is $3r_1/2$; r_1 even
$\overline{2}$ $\overline{5}$ $\overline{6}$ $\overline{2}$ $\overline{6}$ $\overline{5}$	$\frac{1}{2}(6r_5+2)(6r_6 + 3)$	residual is $3r_5/2$; r_5 even
$\overline{4}$ $\overline{3}$ $\overline{4}$ $\overline{4}$ $\overline{4}$ $\overline{3}$	$\frac{1}{2}(6r_3(6r_4+1)$	r_3 odd
$\overline{4}$ $\overline{3}$ $\overline{2}$ $\overline{4}$ $\overline{2}$ $\overline{3}$	$\frac{1}{2}(6r_3(6r_2-1)\ )$	r_3 odd

Table 2

References

1. J.V. Leyendekkers, J.M. Rybak and A.G. Shannon, Integer Class Properties Associated with an integer Matrix. *Notes on Number Theoryand Discrete Mathematics,* 1,2,1995, 53-9

1. J.V. Leyendekkers and J.M. Rybak, The generation and analysis of Pythagorean triples within a two-parameter grid. *International Journal of Mathematical Education in Science and Technology* 26, 6, 1995, 787-93