A PARTIAL DIFFERENCE EQUATION AND A MINIMAL SURFACE

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1. Introduction

Difference equations are often convenient in the mathematical modelling of medical problems because many physiological properties, such as the measurement of plasma glucose levels, tend to be assessed at discrete time intervals. Ollerton and Shannon [4] describe one such linear difference equation in detail in the context of the development of an artificial beta-cell. A partial non-linear difference equation which arose in a similar context is given by

$$(1 + u_{i,j+1}{}^2 - 2u_{i,j}u_{i,j+1} + u_{i,j}{}^2)(u_{i+2,j} - 2u_{i+1,j} + u_{i,j}) - 2((u_{i+1,j} - u_{i,j})u_{i,j+1} - (u_{i+1,j} - u_{i,j})u_{i,j})$$

$$+ (1 + u_{i+i,j}{}^2 - 2u_{i+1,j}u_{i,j} + u_{i,j}{}^2)(u_{i,j+2} - 2u_{i,j+1} + u_{i,j}) = 0. \tag{1}$$

The purpose of this note it to convert this difference equation into the corresponding partial differential equation and to examine the resulting minimal surface. This is done formally without digressing into the issues associated with mapping from the integers to the reals.

2. Analytic Transformation

In order to carry out the transformation we first define two appropriate shift operators:

$$\Delta_i = u_{i+1,j} - u_{i,j}, \tag{2}$$

and

$$\Delta_j = u_{i,j+1} - u_{i,j}. \tag{3}$$

so that (1) can be rewritten as

$$(1 + (\Delta_j u_{i,j})^2)\Delta_i{}^2 u_{i,j} - 2\Delta_i u_{i,j}\Delta_j u_{i,j}\Delta_i\Delta_j u_{i,j} + (1 + (\Delta_i u_{i,j})^2)\Delta_j{}^2 u_{i,j} = 0. \tag{4}$$

This can be transformed into the partial differential equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0 \tag{5}$$

in which the differential operators are represented by

$$p = \partial z/\partial x, \quad q = \partial z/\partial y, \quad r = \partial^2 z/\partial x^2, \quad s = \partial^2 x/\partial x\partial y, \quad t = \partial^2 z/\partial y^2.$$

This is the condition for the mean curvature at (x,y) of the surface $z = f(x,y)$ to be zero and so for the surface to be a minimal surface [3].

3. Geometric Transformation

Few minimal surfaces exist with equations simple enough to permit investigations of their properties. A practical exercise for generating minimal surfaces is to dip a twisted loop of wire in a soap solution as was done by Joseph Plateau (1801-1883). The hyperbolic paraboloid

$$x^2/a^2 - y^2/b^2 = 2cz \tag{6}$$

where a,b,c are constants, is, for small areas, a close approximation to a minimal surface. Another minimal surface is [2]

$$z = ln(cos(kx)/cos(ky))/k \qquad (7)$$

and this is illustrated here.

In fact, the following transformation may be used to give a 2-dimensional representation of a 3-dimensional surface using the top left hand corner of the computer monitor, C, as the origin (**Figure** 1). The angles A,B may be chosen arbitrarily, but $A = 20°$ and $B = 30°$ have been found to be suitable. Appropriate values are $d = 315$ and $f = 100$. The computer and printer were IBM compatible. By considering projections to lines parallel to the sides of the screen, it can be seen that

$$x1 = d + y \cos(B) - x \cos(A) \qquad (8)$$

and

$$y1 = f + y \sin(B) - x \sin(A) - z. \qquad (9)$$

4. Conclusion

The BASIC computer program listed in the appendix is a modification of Barden [1]. The boundaries chosen were $\pi/3 \geq x > 0$ and $\pi/3 > y > 0$. It can be seen that the saddle-shaped surface is such that $P(x,y,z) \longrightarrow P(x1,y1)$. Finally, it is worth noting that this transformation is related to the computation of contour maps of Skala [5].

References

1.   W Barden Jr, *Graphics and Sound for the Tandy 10000s and PC Compatibles*, Microtrend, San Marcos CA, 1988, p.225.

2.   J H Clarke, The hyperbolic paraboloid as an approximation to a surface of minimum area, *Architectural Science Review*, 9(2), 1966: 53-54.

3.   L P Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover, New York, 1966, p.251.

4.   R L Ollerton and A G Shannon, Difference equations and an artificial pancreatic beta-cell, *International Journal of Mathematical Education in Science and Technology*, 22(4), 1991: 545-554.

5.   H Skala, Contour maps - a visual experience, *The College Mathematics Journal*, 22(3), 1991: 241-244.
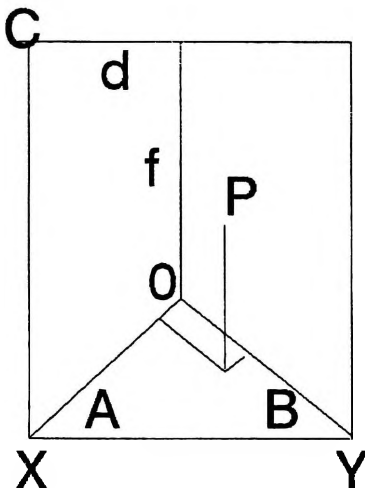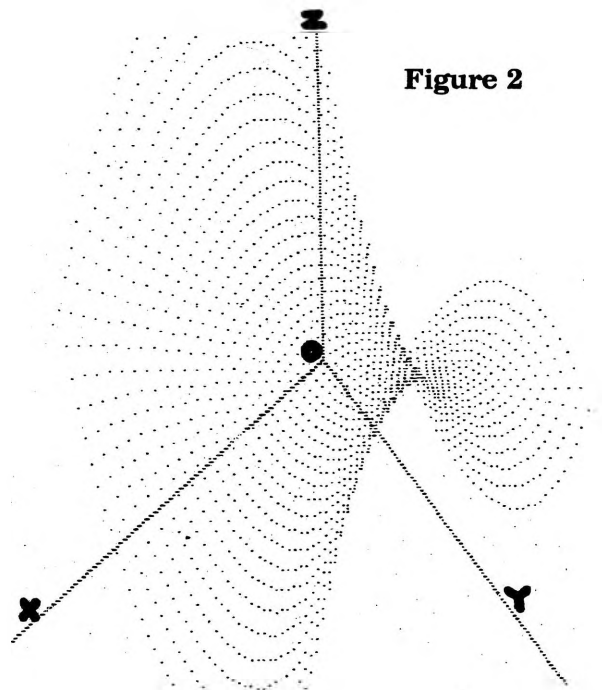
Figure 1

Figure 2

Appendix: Program for Minimal Surface

```
10 CLS :KEY OFF
20 SCREEN 2
30 LINE (315,100)-(0,215)
40 LINE(315,0)-(315,100)
50 LINE(315,100)-(639,287)
60 FOR X=-105 TO 105 STEP 5
70 FOR Y=-105 TO 105 STEP 5
80 K=.01
90 U=LOG(COS(K*X)):V=LOG(COS(K*Y))
100 Z=(U-V)/K
110 X1=315+Y*.866-X*.94
120 Y1=100+Y*.5+X*.342-Z
130 PSET(X1,Y1)
140 NEXT:NEXT
200 LLIST
10000 'GRAPHICS'
10010 DEF SEG=&HB800
10020 OPEN "LPT1:"AS #1
10030 WIDTH"LPT1:",180
10040 PRINT#1,CHR$(27);CHR$(51);CHR$(48);
10050 FOR BYTENO= 10 TO 70
10060 START =80*199+192+BYTENO
10070 PRINT#1,CHR$(13);CHR$(10);
10080 BYTES=START
10090 GOSUB 10160
10100 GOSUB 10190
10110 IF BYTES)=0 AND BYTES <80 THEN 10140
10120 IF BYTES <8000 THEN BYTES =BYTES+8112 ELSE BYTES= BYTES- 8192
10130 GOTO 10100
10140 NEXT BYTENO
10160 'SUBROUTINE'
10170 PRINT#1,CHR$(27);"K";CHR$(144);CHR$(1);
10180 RETURN
10190 'SUBROUTINE'
10200 PRINT#1,CHR$(PEEK(BYTES));CHR$(0);
10210 RETURN
```

**Any surface can be shown
by changing lines 60 to 100**

$$z = \frac{1}{K} \log_e \left(\frac{\cos Kx}{\cos Ky} \right)$$

omit when printing surface

Change 48 to 12 to print surface,
48 for spacing listing.