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NOTE ON TWO INEQUALITIES

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In the paper we shall use the equality $0^0 = 1$.

<u>THEOREM 1</u>: Let $n \geq 2$ be a natural number and $\{x_i\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ be three sequences of nonnegative real numbers. If the conditions: $\sum_{i=1}^n x_i \leq 1$; $\beta := \sum_{i=1}^n \beta_i \leq 1$; $\alpha_i + \beta_i \geq 1$ $(1 \leq i \leq n)$, are satisfied, then the inequality

$$\sum_{i=1}^n x_i^{\alpha_i} . (1 - \sum_{j=1, j\neq i}^n x_j)^{\beta_i} \leq 1 \tag{1}$$

holds.

We shall use the following assertion for the proof of Theorem 1.

LEMMA: For every four nonnegative real numbers u, v, p, q, with $p + q = 1$, the inequality

$$u^p . v^q \leq p.u + q.v \tag{2}$$

holds.

The function $f(x) = e^x$ is convex over the real axis. Therefore,

$$f(p.y + q.z) \leq p.f(y) + q.f(z)$$

for arbitrary nonnegative real numbers y and z. Putting $y = \ln u$, $z = \ln v$, we come to (2), but with positive numbers u and v. When at least one of u and v is equal to zero, (2) is obviously true.

Proof of Theorem 1 considering two cases.

A) $\alpha_i + \beta_i = 1$, for every i $(1 \leq i \leq n)$. In this case we use (2) and obtain

$$x_i^{\alpha_i} . (1 - \sum_{j=1, j\neq i}^n x_j)^{\beta_i} \leq \alpha_i . x_i + \beta_i . (1 - \sum_{j=1, j\neq i}^n x_j), \tag{3}$$

for $1 \leq i \leq n$.

Adding all n inequalities of (3), we get

$$\sum_{i=1}^n x_i^{\alpha_i} . (1 - \sum_{j=1, j\neq i}^n x_j)^{\beta_i} \leq \sum_{i=1}^n \alpha_i . x_i + \sum_{i=1}^n \beta_i . (1 - \sum_{j=1, j\neq i}^n x_j). \tag{4}$$

The right hand of (4) after some elementary computations yields

$$\beta + \sum_{i=1}^{n} (\alpha_i + \beta_i - \sum_{K=1}^{n} \beta_K).x_i . \tag{5}$$

Using the relation $\alpha_i + \beta_i = 1$, for $1 \le i \le n$, we rewrite (5) in the form

$$\beta + (1 - \beta). \sum_{i=1}^{n} x_i . \tag{6}$$

But we obviously have that expression (6) is not greater than 1, because of the $\sum_{i=1}^{n} x_i \le 1$ condition.

B) There exists at least one i_0 for which $\alpha_{i_0} + \beta_{i_0} > 1$.

In this case we consider the numbers α'_i and β'_i $(1 \le i \le n)$, defined by: $\alpha'_i = \dfrac{\alpha_i}{\alpha_i + \beta_i}$ and $\beta'_i = \dfrac{\beta_i}{\alpha_i + \beta_i}$. These numbers satisfy the conditions of the theorem and $\alpha'_i + \beta'_i = 1$ $(1 \le i \le n)$. Then we apply the case A) with α'_i and β'_i, instead of α_i and β_i, and the inequality

$$\sum_{i=1}^{n} x_i^{\alpha'_i}.(1 - \sum_{j=1, j\neq i}^{n} x_j)^{\beta'_i} \le 1 \tag{7}$$

holds. But, obviously,

$$x_i^{\alpha_i} \le x_i^{\alpha'_i}, \quad (1 - \sum_{j=1, j\neq i}^{n} x_j)^{\beta_i} \le (1 - \sum_{j=1, j\neq i}^{n} x_j)^{\beta'_i}$$

for i $(1 \le i \le n)$.

From the last inequalities (1) holds, because of (7).

<u>COROLLARY</u>: If x, y, α, $\beta \ge 0$ are real numbers, $x + y \le 1$, $\alpha + \beta \ge 1$ and $\beta \le 1/2$, then

$$x^{\alpha}.(1 - y)^{\beta} + y^{\alpha}.(1 - y)^{\beta} \le 1 \tag{8}$$

holds.

Obviously, (1) is a generalization of (8), but (8) can be generalized in another direction, too.

<u>THEOREM 2</u>: If x, y, α, $\beta \ge 0$ are real numbers, $x + y \le 1$, $\alpha + \beta \ge 1$ and $\alpha \ge \beta \ge 0$, then

$$x^{\alpha}.(1 - y)^{\beta} + y^{\alpha}.(1 - x)^{\beta} \le 1 \tag{9}$$

holds.

Proof: We shall consider below two cases for $\alpha + \beta$, where by condition $\alpha \geq \beta \geq 0$.

Let $\alpha + \beta = 1$. From (2) it follows that

$$x^{\alpha}.(1 - y)^{\beta} \leq \alpha.x + \beta.(1 - y)$$

and

$$y^{\alpha}.(1 - x)^{\beta} \leq \alpha.y + \beta.(1 - x).$$

Therefore,

$$x^{\alpha}.(1 - y)^{\beta} + y^{\alpha}.(1 - x)^{\beta}$$

$$\leq (\alpha - \beta).(x + y) + 2.\beta \leq \alpha - \beta + 2.\beta \leq \alpha + \beta = 1.$$

Let $\alpha + \beta > 1$. Let us define: $\alpha' = \dfrac{\alpha}{\alpha + \beta}$ and $\beta' = \dfrac{\beta}{\alpha + \beta}$. Obviously, $\alpha' \geq \beta' \geq 0$ and $\alpha' + \beta' = 1$. Thus the first case is applicable and we obtain the inequality

$$x^{\alpha'}.(1 - y)^{\beta'} + y^{\alpha'}.(1 - x)^{\beta'} \leq 1.$$

From $x^{\alpha'} \geq x^{\alpha}$, $(1 - y)^{\beta'} \geq (1 - y)^{\beta}$, $y^{\alpha'} \geq y^{\alpha}$, $(1 - x)^{\beta'} \geq (1 - x)^{\beta}$ it follows that

$$x^{\alpha}.(1 - y)^{\beta} + y^{\alpha}.(1 - x)^{\beta} \leq x^{\alpha'}.(1 - y)^{\beta'} + y^{\alpha'}.(1 - x)^{\beta'} \leq 1.$$

Hence Theorem 2 is valid.

Obviously, if $\beta \leq 1/2$ and $\alpha + \beta \geq 1$, then $\alpha \geq \beta$, i.e., from the validity of (9) follows the validity of (8), but the opposite is not always true.