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ON CERTAIN ARITHMETICAL PRODUCTS INVOLVING REGULAR CONVOLUTIONS

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Abstract. We generalize von Mangoldt's function and certain arithmetical products of trigonometrical functions and Euler's gamma function in terms of Narkiewicz's regular convolutions. We give arithmetic evaluations for these products and we establish asymptotic formulae for them in case of cross-convolutions, investigated by the first author in previous papers.

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1. Introduction

Let A be a regular convolution of Narkiewicz-type ([Nar63]) given by

$$(f *_A g)(n) = \sum_{d\in A(n)} f(d)g(n/d).$$

see also [McC86], [Sit78], [T97-i]. The elements of the set $A(n)$ are called the A-divisors of n. This is a common generalization of the Dirichlet convolution D and of the unitary convolution U, cf. [C60].

Let $\mathbb{N}$ denote the set of positive integers. We recall that if A is a regular convolution, then

(a) for every prime power p^a there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$,

(b) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution, μ_A is multiplicative and for all prime powers p^a one has

$$\mu_A(p^a) = \begin{cases} -1, & \text{if } t_A(p^a) = a, \\ 0, & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{N}$, let $A_k(n) = \{d \in \mathbb{N} : d^k \in A(n^k)\}$. The A_k-convolution is regular whenever the A-convolution is regular, see [Sit78], Theorem 3.1. Let $(a,b)_{A,k}$ denote the largest k-th power divisor of a which belongs to $A(b)$. Note that $(a,b)_{D,1}$ is the usual greatest common divisor of a and b.

In this paper we investigate the arithmetical products

$$P^{(i)}_{S,A,k}(n) = \prod_{\substack{1\le x<n^k \\ ((x,n^k)_{A,k})^{1/k}\in S}} \psi_i\left(\frac{x}{n^k}\right),$$

where $n\in\mathbb{N}, n\ge 2$, S is an arbitrary subset of $\mathbb{N}$, $1\le i\le 3$, $\psi_1(x)=\sin\pi x, \psi_2(x)=(2\sin\pi x)^x$, $\psi_3(x)=\Gamma(x)$ is Euler's gamma function.

We introduce a generalization of von Mangoldt's arithmetical function, we give arithmetical evaluations for these products and establish asymptotic formulae for them in case of cross-convolutions.

The notion of cross-convolution, as a special regular convolution, was introduced and investigated by the first author in [T95],[T97-i], [T-ii], [T-iii] as follows. Let A be a regular convolution. We say that A is a *cross-convolution* if for every prime p we have either $t_A(p^a)=1$, i.e. $A(p^a)=\{1,p,p^2,\dots,p^a\}\equiv D(p^a)$ for every $a\in\mathbb{N}$ or $t_A(p^a)=a$, i.e. $A(p^a)=\{1,p^a\}\equiv U(p^a)$ for every $a\in\mathbb{N}$. Let P and Q be the sets of the primes of the first and second kind of above, respectively, where $P\cup Q=\mathbb{P}$ is the set of all primes. For $P=\mathbb{P}$ and $Q=\emptyset$ we have the Dirichlet convolution D and for $P=\emptyset$ and $Q=\mathbb{P}$ we obtain the unitary convolution U.

Furthermore, let (P) and (Q) denote the multiplicative semigroups generated by $\{1\}\cup P$ and $\{1\}\cup Q$, respectively. Every $n\in\mathbb{N}$ can be written uniquely in the form $n=n_Pn_Q$, where $n_P\in(P), n_Q\in(Q)$.

Remark 1. If A is a cross-convolution, then $A_k=A$ for every $k\in\mathbb{N}$, see [Sit78], Theorem 3.3.

Our results generalize earlier results of [ST90], given in the case $A=D$ and $k=1$, see also [ST89].

2. Arithmetical evaluations

For a subset S of $\mathbb{N}$ let ρ_S denote the characteristic function of S, that is $\rho_S(n)=1$ if $n\in S$, and $\rho_S(n)=0$ if $n\notin S$. The generalized Möbius function $\mu_{S,A}$ is defined by

$$\sum_{d\in A(n)} \mu_{S,A}(d)=\rho_S(n),\quad n\in\mathbb{N}, \tag{1}$$

see P. Haukkanen [H88]. If $S=\{1\}$, then $\mu_{S,A}=\mu_A$, and if $A=D$, then $\mu_A=\mu$ is the classical Möbius function. The function $\mu_{S,D}$ was introduced by E. Cohen [C59].

We say that S is multiplicative if its characteristic function ρ_S is multiplicative, i.e. $1\in S$ and $mn\in S$ if and only if $m\in S, n\in S$ for every $m,n\in\mathbb{N}$ with $(m,n)=1$.

In order to give arithmetical evaluations of the above products we introduce the following generalizations of von Mangoldt's function. For a regular convolution A let

$$\Lambda_A(n)=\sum_{d\in A(n)}\mu_A(n/d)\log d=-\sum_{d\in A(n)}\mu_A(d)\log d, \tag{2}$$

for every $n \in \mathbb{N}$ and if $S \subseteq \mathbb{N}$, let

$$(3) \qquad \Lambda_{S,A}(n) = \sum_{d \in A(n)} \rho_S(d)\Lambda_A(n/d),$$

for every $n \in \mathbb{N}$.

If $A = D$, then $\Lambda_{S,D} \equiv \Lambda_S$ is the function investigated by us in [ST90]. If $S = \{1\}$, then $\Lambda_{\{1\},A} \equiv \Lambda_A$ and if $S = \mathbb{N}$, then $\Lambda_{\mathbb{N},A}(n) = \log n$, for every $n \in \mathbb{N}$ and for every regular convolution A. Λ_D is the classical function of von Mangoldt, $\Lambda_U \equiv \Lambda^*$ was defined by A. BEGE [B90].

Theorem 1. *If A is a regular convolution, $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, then*

$$\text{(i)} \qquad \sum_{d \in A(n)} \Lambda_A(d) = \log n,$$

$$\text{(ii)} \qquad \Lambda_A(n) = \begin{cases} t \log p, & \text{if } n = p^a, \text{ a prime power and } t = t_A(p^a), \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{(iii)} \qquad \Lambda_{S,A}(n) = \sum_{d \in A(n)} \mu_{S,A}(n/d) \log d,$$

$$\text{(iv)} \qquad \Lambda_{S,A}(n) = \log \prod_{i=1}^{r} p_i^{c_i},$$

where $n = \prod_{i=1}^{r} p_i^{a_i}$, $t_i = t_A(p_i^{a_i})$ *and*

$$c_i = t_i \sum_{j=1}^{a_i/t_i} \rho_S(n/p_i^{j t_i}), \quad i \in \{1, 2, ..., r\}.$$

Proof. (i) According to (2) we have $\Lambda_A = \log *_A \mu_A$, therefore $\Lambda_A *_A I = \log *_A \mu_A *_A I = \log$.

(ii) If $n = p^a$ is a prime power and $t = t_A(p^a)$, then

$$\Lambda_A(p^a) = -\log 1 - \mu_A(p^t) \log p^t = t \log p,$$

using the properties of μ_A. If $n = \prod_{i=1}^{r} p_i^{a_i}$, with $r \geq 2$ and $t_i = t_A(p_i^{a_i})$, then we obtain from (2)

$$\Lambda_A(n) = -\Bigg(\log 1 + \sum_{i=1}^{r} \mu_A(p_i^{t_i}) \log p_i^{t_i} + \sum_{1 \leq i < j \leq r} \mu_A(p_i^{t_i} p_j^{t_j}) \log(p_i^{t_i} p_j^{t_j})$$

$$+ \sum_{1 \leq i < j < k \leq r} \mu_A(p_i^{t_i} p_j^{t_j} p_k^{t_k}) \log(p_i^{t_i} p_j^{t_j} p_k^{t_k}) + ... + \mu_A(p_1^{t_1} ... p_r^{t_r}) \log(p_1^{t_1} ... p_r^{t_r})\Bigg)$$

$$= \sum_{i=1}^{r} \log p_i^{t_i} - \sum_{1\le i<j\le r} \log(p_i^{t_i} p_j^{t_j}) + \sum_{1\le i<j<k\le r} \log(p_i^{t_i} p_j^{t_j} p_k^{t_k}) - ... + (-1)^r \log(p_1^{t_1} ... p_r^{t_r})$$

$$= \log \left((\prod_{i=1}^{r} p_i^{t_i})(\prod_{1\le i<j\le r} p_i^{t_i} p_j^{t_j})^{-1} (\prod_{1\le i<i<k\le r} p_i^{t_i} p_j^{t_j} p_k^{t_k}) ... (p_1^{t_1} ... p_r^{t_r})^{(-1)^r} \right).$$

For every $i \in \{1, 2, ..., r\}$, the exponent of $p_i^{t_i}$ is

$$1 - \binom{r-1}{1} + \binom{r-2}{2} - ... + (-1)^r \binom{r-1}{r-1} = 0,$$

which finishes the proof of (ii).

(iii) Using (3), (2) and (1) we get

$$\Lambda_{S,A} = \rho_S *_A \Lambda_A = \rho_S *_A \log *_A \mu_A = \log *_A \mu_{S,A}.$$

(iv) From (3) and (ii) we obtain

$$\Lambda_{S,A}(n) = \sum_{d\in A(n)} \Lambda_A(d) \rho_S(n/d) = \sum_{i=1}^{r} \sum_{j=1}^{a_i/t_i} \Lambda_A(p_i^{jt_i}) \rho_S(n/p_i^{jt_i})$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{a_i/t_i} (t_i \log p_i) \rho_S(n/p_i^{jt_i}) = \log \prod_{i=1}^{r} p_i^{c_i}.$$

For a regular convolution A, for $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ let $\phi_{S,A,k}(n)$ denote the number of integers $x \pmod{n^k}$ such that $((x, n^k)_{A,k})^{1/k} \in S$. This generalized Euler function was introduced by P. HAUKKANEN [H88] and one has

(4) $$\phi_{S,A,k} = \mu_{S,A_k} *_{A_k} E_k,$$

where $E_k(n) = n^k, n \in \mathbb{N}$. Observe that $\phi_{\{1\},D,1} \equiv \phi$ is Euler's arithmetical function. For other special cases investigated in the literature see [H88], [T97-i].

Theorem 2. *If A is regular convolution, $S \subseteq \mathbb{N}$ and $k, n \in \mathbb{N}, n \ge 2$, then*

(5) $$P^{(1)}_{S,A,k}(n) = \exp k\Lambda_{S,A_k}(n) / 2^{\phi_{S,A,k}(n) - \rho_S(n)},$$

(6) $$P^{(2)}_{S,A,k}(n) = \exp \frac{k}{2} \Lambda_{S,A_k}(n),$$

(7) $$P^{(3)}_{S,A,k}(n) = \left((2\pi)^{\phi_{S,A,k}(n) - \rho_S(n)} / \exp k\Lambda_{S,A_k}(n) \right)^{1/2}.$$

Proof. Applying the fact that $d^k \in A((a,b)_{A,k})$ if and only if $d^k|a$ and $d^k \in A(b)$, see [Sit78], Theorem 4.2, we have

$$\log P^{(i)}_{S,A,k}(n) = \sum_{\substack{1\le x<n^k \\ ((x,n^k)_{A,k})^{1/k}\in S}} \log \psi_i(x/n^k) = \sum_{1\le x<n^k} \log \psi_i(x/n^k)\rho_S(((x,n^k)_{A,k})^{1/k})$$

$$= \sum_{1\le x<n^k} \log \psi_i(x/n^k)\left(\sum_{d^k\in A((x,n^k)_{A,k})} \mu_{S,A_k}(d)\right)$$

$$= \sum_{1\le x<n^k} \log \psi_i(x/n^k)\left(\sum_{\substack{d\in A_k(n)\\ d^k|x}} \mu_{S,A_k}(d)\right)$$

$$= \sum_{d\in A_k(n)} \mu_{S,A_k}(d) \sum_{1\le y=x/d^k<(n/d)^k} \log \psi_i(yd^k/n^k).$$

Hence

$$\log P^{(i)}_{S,A,k}(n) = \sum_{d\in A_k(n)} \mu_{S,A_k}(d) \log \prod_{1\le y<(n/d)^k} \psi_i(y/(n/d)^k). \tag{8}$$

Now let $\psi_1(x) = \sin \pi x$. Then using the well-known formula

$$\prod_{1\le x<n} \sin\frac{\pi x}{n} = n/2^{n-1}, \quad n\in\mathbb{N}, n\ge 2,$$

we obtain from (8)

$$\log P^{(1)}_{S,A,k}(n) = \sum_{d\in A_k(n)} \mu_{S,A_k}(d)\log((n/d)^k/2^{(n/d)^k-1})$$

$$= k\sum_{d\in A_k(n)} \mu_{S,A_k}(d)\log(n/d) - (\log 2)\left(\sum_{d\in A_k(n)} \mu_{S,A_k}(d)(n/d)^k - \sum_{d\in A_k(n)} \mu_{S,A_k}(d)\right)$$

$$= k\Lambda_{S,A_k}(n) - (\log 2)(\phi_{S,A,k}(n) - \rho_S(n)),$$

by Theorem 1/(iii), (1) and (4), and we get formula (5).

Now for $\psi_2(x) = (2\sin\pi x)^x$ and applying the formula

$$\prod_{1\le x<n} (2\sin\frac{\pi x}{n})^x = n^{n/2},$$

see B. Malvina [M86], we get from (8)

$$\log P^{(2)}_{S,A,k}(n) = \sum_{d\in A_k(n)} \mu_{S,A_k}(d)\log(n/d)^{k/2} = \frac{k}{2}\Lambda_{S,A_k}(n),$$

which is formula (6). Finally, for $\psi_3(x) = \Gamma(x)$ and taking into account of

$$\prod_{1\le x<n} \Gamma(x/n) = ((2\pi)^{n-1}/n)^{1/2},$$

we obtain

$$\log P^{(3)}_{S,A_k}(n) = \frac{1}{2} \sum_{d\in A_k(n)} \mu_{S,A_k}(d) \log((2\pi)^{(n/d)^k-1}/(n/d)^k)$$

$$= \frac{1}{2}(\log 2\pi(\phi_{S,A,k}(n) - \rho_S(n)) - k\Lambda_{S,A_k}(n)),$$

which gives (7).

3. Asymptotic formulae

We need the following lemmas

Lemma 1. ([T97-i], Theorem 8) *If A is a cross-convolution, $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, then*

$$\sum_{n\le x} \phi_{S,A,k}(n) = \frac{\beta_{S,A,k} x^{k+1}}{k+1} + O(C_k(S,x,Q)),$$

where

$$\beta_{S,A,k} = \sum_{n=1}^{\infty} \frac{\mu_{S,A}(n)\phi(n_Q)}{n^{k+1}n_Q}$$

and $C_k(S,x,Q) = x^k$ $(k>1)$, $x\log^4 x$ $(k=1$ and Q infinite set$)$, $x\log^2 x$ $(k=1$ and Q finite set or $k=1, Q$ infinite set and S multiplicative$)$, $x \log x$ $(k=1, Q$ finite set and S multiplicative$)$.

Lemma 2. *For every $S \subseteq \mathbb{N}$ and for every regular convolution A,*

$$\sum_{n\le x} \Lambda_{S,A}(n) = O(x\log x).$$

Proof. From Theorem 1/(ii) we have $\Lambda_A(n) \ge 0$ for every $n \in \mathbb{N}$ and by (3)

$$0 \le \Lambda_{S,A}(n) \le \sum_{d\in A(n)} \Lambda_A(n/d) = \log n,$$

therefore

$$\sum_{n\le x} \Lambda_{S,A}(n) \le \sum_{n\le x} \log n = O(x\log x).$$

Remark 2. It is well-known, that for von Mangoldt's function one has $\sum_{n\le x}\Lambda(n) = O(x)$. If S is such that the Dirichlet series of ρ_S is convergent in $z=1$ (in particular if S is finite), then for the function $\Lambda_{S,D} \equiv \Lambda_S$ we get $\sum_{n\le x}\Lambda_S(n) = O(x)$. Indeed, by (3) we can deduce

$$\sum_{n\le x} \Lambda_S(n) = \sum_{n\le x}\sum_{d|n} \rho_S(d)\Lambda(n/d) = \sum_{d\le x}\rho_S(d) \sum_{e\le x/d} \Lambda(e)$$

$$= \sum_{d\le x} \rho_S(d) O(x/d) = O(x \sum_{d\le x} \frac{\rho_S(d)}{d}) = O(x).$$

Theorem 3. *If A is a cross-convolution, $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, then*

$$\sum_{n \le x} \log P^{(1)}_{S,A,k}(n) = -\frac{\log 2}{k+1} \beta_{S,A,k} x^{k+1} + O(C_k(S,x,Q)), \tag{9}$$

$$\sum_{n \le x} \log P^{(2)}_{S,A,k}(n) = O(x \log x), \tag{10}$$

$$\sum_{n \le x} \log P^{(3)}_{S,A,k}(n) = \frac{\log 2\pi}{2(k+1)} \beta_{S,A,k} x^{k+1} + O(C_k(S,x,Q)), \tag{11}$$

where $\beta_{S,A,k}$ and $C_k(S,x,Q)$ are given in Lemma 1.

Proof. By Remark 1, Theorem 2, Lemma 1 and Lemma 2 we obtain

$$\sum_{n \le x} \log P^{(1)}_{S,A,k}(n) = \sum_{n \le x} (k\Lambda_{S,A}(n) - (\phi_{S,A,k}(n) - \rho_S(n)) \log 2)$$

$$= k \sum_{n \le x} \Lambda_{S,A}(n) - (\log 2)(\sum_{n \le x} \phi_{S,A,k}(n) - \sum_{n \le x} \rho_S(n))$$

$$= O(x \log x) - (\log 2)(\frac{1}{k+1} \beta_{S,A,k} x^{k+1} + O(C_k(S,x,Q)) + O(x))$$

$$= -\frac{1}{k+1} \beta_{S,A,k} (\log 2) x^{k+1} + O(C_k(S,x,Q)),$$

which is (9). Furthermore,

$$\sum_{n \le x} \log P^{(3)}_{S,A,k}(n) = \sum_{n \le x} \frac{1}{2}((\phi_{S,A,k}(n) - \rho_S(n)) \log 2\pi - k\Lambda_{S,A}(n))$$

$$= \frac{\log 2\pi}{2}(\frac{1}{k+1} \beta_{S,A,k} x^{k+1} + O(C_k(S,x,Q)) + O(x) + O(x \log x),$$

and we get (11), while (10) yields at once by Lemma 2.

Corollary. ($S = \{1\}$) *For $P^{(i)}_{\{1\},A,k} \equiv P^{(i)}_{A,k}, i \in \{1,3\}$ we have*

$$\sum_{n \le x} \log P^{(1)}_{A,k}(n) = -\frac{\log 2}{k+1} \beta_{A,k} x^{k+1} + O(D_k(x,Q)),$$

$$\sum_{n \le x} \log P^{(3)}_{A,k}(n) = \frac{\log 2\pi}{2(k+1)} \beta_{A,k} x^{k+1} + O(D_k(x,Q)),$$

where

$$\beta_{A,k} = \prod_{p \in P} \left(1 - \frac{1}{p^{k+1}}\right) \prod_{p \in Q} \left(1 - \frac{p-1}{p(p^{k+1}-1)}\right)$$

and $D_k(x,Q) = x^k$ ($k > 1$), $x \log^2 x$ ($k = 1$ and Q infinite set), $x \log x$ ($k = 1$ and Q finite set).

Proof. Use Theorem 3 and apply Euler's product formula for the series $\beta_{\{1\},A,k}$, cf. [T97-i], Lemma 10.

The particular case $A = D, k = 1$ was considered in our papers [ST89], [ST90].

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