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ANALYSIS OF ODD EXPONENT TRIPLES WITHIN THE MODULAR RING $\mathbb{Z}_4$ USING BINOMIAL EXPANSIONS AND FERMAT REDUCTIONS

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ABSTRACT

The essential characteristics of integers and the relationships with their powers are explored within the framework of the modular ring $\mathbb{Z}_4$ in order to analyse why odd powered triples with exponents greater than unity cannot exist in integer form. Two methods are given which exploit old expansion and reduction techniques in a new way. By way of conclusion the second method is also illustrated by reference to Pythagorean triples.

1. INTRODUCTION

In this paper we analyse why odd power integer triples a, b, c, $\epsilon\, \mathbb{Z}_4$:

$$c^n = b^n + a^n \tag{1.1}$$

do not exist. We utilise techniques which depend on the properties of the integers themselves rather than complex functions or as solutions in fields $\mathbb{Q}_b$ of p-adic rationals [5]. We characterise the integers with the classes $\overline{0}$, $\overline{1}$, $\overline{2}$, $\overline{3}$ of the modular ring $\mathbb{Z}_4$ within which the integers N are given by

$$N = 4R_i + i \tag{1.2}$$

where i represents the class and R_i the row within i [3]. Since there is no $N : N^n \epsilon \{\overline{2}\}$, n odd, there are only three classes which contain odd powers.

It was also shown in [4] that we need only consider the class sets $< \overline{0}\,\overline{1}\,\overline{3} >$, $< \overline{2}\,\overline{1}\,\overline{3} >$, $< \overline{1}\,\overline{2}\,\overline{1} >$ and $< \overline{1}\,\overline{0}\,\overline{1} >$ since the components b, a may be interchanged for the present analysis.

Two different types of analysis are given, both using $\mathbb{Z}_4$. The first method relies on the simple binomial expansion and the remainder theorem, and uses a theorem of Fermat [1] which show that for prime $p\,(N,p)=1$ and $k_N \epsilon \mathbb{Z}$

$$N^{p-1} = 1 + pk_N \tag{1.3}$$

The second method exploits the extended theorem of Fermat.

$$N^p = N + pK_N \tag{1.4}$$

The usefulness of equation (1.4) is increased by analysing K_N within the framework of $\mathbb{Z}_4$. This gives the analysis a great deal of flexibility by allowing powers to be reduced in order to simplify equations in a chosen direction. The apparently simple equation (1.1), which has intrigued and frustrated mathematicians for centuries [5], emerges thus as a sort of Dr Who's police box of inner complexities.

2. METHOD 1

This method looks at the binomial expansion structure for the powers and deduces the constraints required before integer solutions can be expected. In obtaining the constraints, the $\mathbb{Z}_4$ class structure, the parities and prime characteristics of the integers are taken into account. This detail for the different quantities and relationships makes the impossibility of finding integer solutions more understandable.

2.1 CLASS SET $\bar{0}\ \bar{1}\ \bar{3}$

Here we have

$$(4R_0)^n = (4R_1+1)^n + (4R_3+3)^n \tag{2.1}$$

for which there should be no integer solutions.

To understand why, equation (2.1) is expanded and divided throughout by 4, to give

$$4^{n-1}(R_o^n - R_1^n - R_3^n) - A = 4nf(R_1,R_3) + n(R_1 + 3^{n-1}R_3). \tag{2.2}$$

The coefficient A is odd and prime to n and is given by

$$A = (1+3^n)/4. \tag{2.3}$$

For example, for $n = 3, 5\,\text{or}\ 7$, $A = 7, 61\,\text{or}\ 547$.

From the parity considerations of equation (2.2) R_1 and R_3 must have opposite parity. The function $f(R_1, R_3)$ is given by

$$f(R_1, R_3) = \sum_{j=1}^{n-2} \frac{n_j}{n} 4^{n-j-2}(R_1^{n-j} + 3^j R_3{}^{n-j})/j! \quad (2.4)$$

in which n_j is the falling factorial coefficient

$$n_j = n(n-1)\dots(n-j+1).$$

Some examples are given in Table 2.1.

Equation (2.2) indicates that, firstly, a residual of A/n will be obtained unless the first term on the LHS is prime to n, and secondly, that the LHS must be divisible by n. The first requirement is considered by using two forms of equation (1.3).

Since

$$R_0^n - R_1{}^n - R_3{}^n = R_o - R_1 - R_3 + nQ_{013} \quad (2.5)$$

where Q_{013} is an even integer, the requirement is that $(R_0 - (R_1 + R_3))$ is prime to n : thus

$$R_o - (R_1 + R_3) = (1 + nK_{013})^{1/(n-1)} \quad (2.6)$$

where K_{013} is an integer.

We shall consider the different parities of R_0 separately.

R_0 ODD

In this case the LHS of equation (2.6) is even and K_{013} must be odd.

For integer solutions of equation (2.6),

$$R_0 - (R_1 + R_3) = d + 2nw \quad (2.7)$$

with $w = 0, 1, 2, 3\dots$ and $d = 2, 4, 6, 8, \dots 2(n-1)$.

Substitution of equations (2.5) and (2.7) into equation (2.2) and division by n gives:

$$(4^{n-1}d - A)/n + 4^{n-1}(2w + Q_{013}) = 4f(R_1, R_3) + (R_1 + 3^{n-1}R_3) \quad (2.8)$$

For most values of d, $(4^{n-1}d - A)$ is prime to n and no integer solution is possible. In those cases where $(4^{n-1}d - A)/n$ is integer, $d = (n+1)$. For these cases we need to examine the constraints on R_1 and R_3 as follows.

Equation (2.1) may be put in the form

$$4^{n-1}R_0{}^n = (R_1 + R_3 + 1)g(R_1, R_3) \quad (2.9)$$

where

$$g(R_1, R_3) = \sum_{j=0}^{n-1} (-1)^j x^{n-j-1} y^j,$$

with

$$x = (4R_1 + 1) \text{ and } y = (4R_3 + 3).$$

Since $g(R_1, R_3)$ has n terms and x and y are both odd, the value of this function is always odd. The requirement then is that $(R_1 + R_3 + 1)/4^{n-1}$ is an integer. Obviously, suitable components would have to be very large as n becomes very large. For integer solutions, $(R_1 + R_3)$ would need to have the form:

$$R_1 + R_3 = (4^{n-1} - 1) + 2 \times 4^{n-1} m \tag{2.10}$$

with $m = 0, 1, 2, 3 \ldots$

Equation (2.9) gives the class couplings of $< \overline{1}, \overline{2} >$ or $< \overline{0}, \overline{3} >$ for both (R_1, R_3) and (R_3, R_1). See the parity analysis for w and m below.

All the above constraints prevent any integer solutions.

Parities of w and m.

If R_0 is in class $\overline{3}$, $R_0 = 4r_3' + 3$ and

$$R_0 = R_1 + R_3 + n(2w + 1) + 1 \tag{2.11}$$

from equation (2.7) with $d = (n + 1)$, and, from equation (2.10),

$$R_0 = 4^{n-1} + 2 \times 4^{n-1} m + n(2w + 1). \tag{2.12}$$

Thus, $$r'_3 = 4^{n-2}(2m + 1) + nw/2 + (n - 3)/4. \tag{2.13}$$

Substitution of the appropriate n gives the parities listed in Table 2.2.

If R_0 is in class $\overline{1}$, $R_0 = 4r_1' + 1$ and

$$r_1' = 4^{n-2}(1 + 2m) + nw/2 + (n - 1)/4. \tag{2.14}$$

Values of the parities of w for this case are also listed in Table 2.2.

To assess the parity of m, equation (2.10) is substituted into equation (2.9) to give

$$R_0{}^n = (1 + 2m) g(R_1, R_3). \tag{2.15}$$

If $A/4$ has a remainder of $3/4$ then $g(R_1, R_3)$ falls in class $\overline{3}$. If $A/4$ has a remainder of $1/4$ then $g(R_1, R_3)$ falls in class $\overline{1}$. Table 2.2 shows the compatible classes for R_0 and $(1 + 2m)$.

In all cases m and w have the same parity.

As an example, consider $n = 3$.

Equation (2.8) becomes, with the coefficient equal to $[4^n(1+n) - (1+3^n)]/4n$,

$$19 + 32w + 16Q_{013} = 4(R_1 + R_3)^2 + (R_1 + R_3) + 8({R_3}^2 + R_3 - R_1R_3), \qquad (2.16)$$

where $4(R_1 + R_3)^2 = 64 \times 4(16m^2 + 15m) + 4 \times 15^2$ and $(R_1 + R_3) = 15 + 32m$.

With $R_1 = 4r_1 + 1, R_3 = 4r_2 + 2,$ then

$$8({R_3}^2 + R_3 - R_1R_3) = 64(2{r_2}^2 + 2r_2 - 2r_2r_1 - r_1) + 32.$$

Collection of the numerical terms yields a coefficient of 928 which is not divisible by the common factor 64; $Q_{013}/4$ is an integer and $(w - m)$ is even (Table 2.2). Hence there is no integer solution when R_1 is in class $\bar{1}$ and R_3 is in class $\bar{2}$. Table 2.3 lists the other class combinations possible for R_1 and R_3. Each combination changes the numerical coefficient and $8({R_3}^2 + R_3 - R_1R_3)$ but not the other terms. A fractional residual is obtained in each case so that no integer solutions are possible when R_0 is odd.

n	$f(R_1, R_3)$	A
3	$({R_1}^2 + 3{R_3}^2)$	7
5	$[16({R_1}^4 + 3{R_3}^4) + 8({R_1}^3 + 9{R_3}^3) + 2({R_1}^2 + 3^3{R_3}^2]$	61
7	$[4^4({R_1}^6 + 3{R_3}^6) + 4^3 \times 3({R_1}^5 + 3^2{R_3}^5) + 5 \times 4^2({R_1}^4 + 3^3{R_3}^4)$ $+ 5 \times 4({R_1}^3 + 3^4{R_3}^3) + 3({R_1}^2 + 3^5{R_3}^2)]$	547

TABLE 2.1

n		Classes		Parities	
	$g(R_1, R_3)$	$(1 + 2m)$	R_0	m	w
$3, 7, 11, \ldots$	$\bar{3}$	$\bar{1}$	$\bar{3}$	even	even
Class $\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	odd	odd
$5, 13, 17, \ldots$	$\bar{1}$	$\bar{3}$	$\bar{3}$	odd	odd
Class $\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	even	even

TABLE 2.2

R_1	R_3	$8({R_3}^2 + R_3 - R_1R_3)*$	Coefft	q	Residual (Coefft/q)
$4r_1+1$	$4r_2+2$	$64(2{r_2}^2 + 2r_2 - 2r_2r_1 - r_1) + 32$	928	64	29/2
$4r_2+2$	$4r_1+1$	$64(2{r_1}^2 - r_2 - 2r_1r_2 + 4m) + 96$	992	64	31/2
$4r_0$	$4r_3+3$	$64(2{r_3}^2 - 12m + 5r_3 - 2r_0r_3) - 192$	704	128	11/2
$4r_3+3$	$4r_0$	$8 \times 16({r_0}^2 - r_0r_3) - 64r_0$	832	128	13/2**

TABLE 2.3

* when m appears the quantity $(r_i + r_j) = 3 + 8m$ has been substituted (see equation 2.10);

** $(w - m)$ is in class $\overline{2}$ and $(Q_{013}/4 + r_0)$ is even.

R_o EVEN

This case is simpler since only equations (2.9) and (2.10) need be considered.

Because $(R_1 + R_3 + 1)/4^{n-1}$ must be even, then for integer solutions

$$R_1 + R_3 = (2 \times 4^{n-1} - 1) + 2 \times 4^{n-1}m \quad (2.17)$$

and equation (2.9) becomes

$$R_0^n = 2(1+m)g(R_1, R_3). \quad (2.18)$$

If R_0 is in class $\overline{0}$, then

$$R_0 = 4r_0' \quad (2.19)$$

so that

$${r_0'}^n = 2(1+m)g(R_1, R_3)/4^n. \quad (2.20)$$

If r_0' is odd, then the case reverts back to that discussed above, for which there are no integer solutions. If r_0' is even, then the denominator of equation (2.20) increases again and the analysis repeats along the same lines, so that a non-integer loop occurs. When R_0 is in class $\overline{2}$,

$$R_0 = 4r_2' + 2, \quad (2.22)$$

and

$$(2r_2' + 1)^n = 2(1+m)g(R_1, R_3)/2^n, \quad (2.23)$$

so that we revert directly to the odd R_0 case.

An even R_0 changes the parameters of equations (2.6) and (2.7) in an interesting way. K_{013} must now be even, which gives $d = 1, n_1, n_2, \ldots (2n-1)$, with $n_i \neq n$. For example, for $n = 7$, $d = 1, 3, 5, 9, 11, 13$. However, only $d = 1$ makes the

coefficient of equation (2.8) integral. This integer coefficient will now have the form $(4^n - (1+3^n))/4n$. Imposed on these restraints is the additional one that $2(1+m)/4^n$ must be an integer.
As for R_0 odd, we can easily deduce that

$$r_0{}' = 2 \times 4^{n-2}(1+m) + nw/2 \tag{2.24}$$

and

$$r_2{}' = 2 \times 4^{n-2}(1+m) + (nw-1)/2 \tag{2.25}$$

which gives the necessary parities of w for integer solutions. Since the parity of R_0 does not affect g(R_1, R_3), and since $R_0{}^n$ can only fall in class $\overline{0}$ (class $\overline{2}$ having no numbers raised to a power), then $(1+m)$ must fall in class $\overline{2}$ or $\overline{0}$ and so m is always odd (Table 2.4).

n	Classes			Parities	
	$g(R_1, R_3)$	$(1+m)$	R_0	m	w
$3, 7, 11, \ldots$	$\overline{3}$	$\overline{2}$ or $\overline{0}$	$\overline{0}$	odd	even
	$\overline{3}$	$\overline{2}$ or $\overline{0}$	$\overline{2}$	odd	odd
$5, 13, 17, \ldots$	$\overline{1}$	$\overline{2}$ or $\overline{0}$	$\overline{0}$	odd	even
	$\overline{1}$	$\overline{2}$ or $\overline{0}$	$\overline{2}$	odd	odd

TABLE 2.4

2.2 CLASS SET $\overline{2}\,\overline{1}\,\overline{3}$

For the set $\overline{2}\,\overline{1}\,\overline{3}$, we have

$$(4R_2+2)^n = (4R_1+1)^n + (4R_3+3)^n \tag{2.26}$$

which, when expanded and divided by $4n$ gives :

$$(4^{n-1}(R_2{}^n - R_1{}^n - R_3{}^n) - A)/n + 4f(R_2, R_1, R_3) +$$
$$(2^{n-1}R_2 - R_1 - 3^{n-1}R_3) = 0 \tag{2.27}$$

with $$A = -(2^n - 1^n - 3^n)/4 \tag{2.28}$$

and $$f(R_2, R_1, R_3) = 4^{n-3}(2R_2{}^{n-1} - R_1{}^{n-1} + 3R_3{}^{n-1}) +$$
$$\tfrac{1}{2}(n-1)\sum_{j=2}^{n-2} 4^{n-j-2}(2^j R_2{}^{n-j} - R_1{}^{n-j} - 3^j R_3{}^{n-j}) \tag{2.29}$$

Using Fermat's theorem, we obtain

$$R_2{}^n - R_1{}^n - R_3{}^n = R_2 - R_1 - R_3 + nQ_{213} \tag{2.30}$$

with Q_{213} an even integer. If $(R_2 - R_1 - R_3)$ is prime to n, then for the set $\overline{2}\,\overline{1}\,\overline{3}$, we get

$$R_2 - (R_1 + R_3) = (1 + nK_{213})^{1/(n-1)}. \tag{2.31}$$

R_2 ODD

The left hand side of equation (2.31) is even (R_1 and R_3 having opposite parity), so that K_{213} is odd. Thus, for integer solutions of equation (2.31)

$$R_2 - (R_1 + R_3) = d + 2nw \tag{2.32}$$

with w and d being defined the same as for equation (2.7).

Equation (2.27) becomes:

$$(4^{n-1}d - A)/n + 4^{n-1}(2w + Q_{213}) + 4f(R_2, R_1, R_3) + 2^{n-1}R_2 -$$
$$(3^{n-1} - 1)R_3 - (R_1 + R_3) = 0 \tag{2.33}$$

Some permissible values of A and the corresponding integer coefficient of equation (2.33) (first term on the LHS) are listed in Table 2.5.

n	d	A	Coefficient
3	2	5	9
5	8	53	399
7	4	515	2267
11	6	43775	567971
13	20	396533	25780599

TABLE 2.5

For $n = 3, 7, 11\ldots, d = (n+1)/2$, but for the series for which $(n+1)/2$ is odd, (n in class $\bar{1}$), $d = (3n+1)/2$.

Equation (2.26) can be put in the form

$$(2R_2 + 1)^n = \Big((R_1 + R_3 + 1)/2^{n-2}\Big)g(R_1, R_3). \tag{2.34}$$

For integer solutions $(R_1 + R_3 + 1)/2^{n-2}$ must be both an integer and odd. This condition is satisfied by:

$$R_1 + R_3 = (2^{n-2} - 1) + 2^{n-1}m. \tag{2.35}$$

Thus, $(R_1 + R_3 + 1)/2^{n-2}$ is given by $(1 + 2m)$ so that, for $n = 3$, (R_1, R_3) or (R_3, R_1) must fall in the two class sets $< \bar{1}, \bar{0} >$ or $< \bar{3}, \bar{2} >$. This gives $m = (r_1 + r_0)$ for the first set and $m = (r_3 + r_2) + 1$ for the second set. However, for $n > 3$, (R_1, R_3) or (R_3, R_1) must fall in the classes $< \bar{3}, \bar{0} >$ or $< \bar{1}, \bar{2} >$. (Table 2.6 gives the corresponding values for m.)

R_2 will be in class $\bar{1}$ or $\bar{3}$. In both cases $(2R_2 + 1)$ falls in class $\bar{3}$ so that, if $g(R_1, R_3)$ is in class $\bar{3}$ $(n = 3, 7, 11\ldots)$, then $(1 + 2m)$ falls in class $\bar{1}$ and m is even.

If $g(R_1, R_3)$ is in class $\bar{1}$ $(n = 5, 13, 17, \ldots)$, then $(1 + 2m)$ falls in class $\bar{3}$ and m is odd.

n	R_3 or R_1 class	R_1 or R_3 class	m
3	$\bar{1}$	$\bar{0}$	$(r_1 + r_0)$
	$\bar{3}$	$\bar{2}$	$(r_3 + r_2) + 1$
5	$\bar{1}$	$\bar{2}$	$(r_1 + r_2 - 1)/4$
	$\bar{3}$	$\bar{0}$	$(r_3 + r_0 - 1)/4$
7	$\bar{1}$	$\bar{2}$	$(r_1 + r_2 - 7)/16$
	$\bar{3}$	$\bar{0}$	$(r_3 + r_0 - 7)/16$
11	$\bar{1}$	$\bar{2}$	$(r_1 + r_2 - 127)/256$
	$\bar{3}$	$\bar{0}$	$(r_3 + r_0 - 127)/256$
13	$\bar{1}$	$\bar{2}$	$(r_1 + r_2 - 511)/1024$
	$\bar{3}$	$\bar{0}$	$(r_3 + r_0 - 511)/1024$

TABLE 2.6

All the above constraints prevent any integer solutions. For example, when $n = 7$ equation (2.33) becomes

$$2267 + 4^6(2w + Q_{213}) + 4f(R_2, R_1, R_3,) + 2^6 R_2 - 728R_3 - (R_1 + R_3) = 0. \quad (2.36)$$

Substituting in $(R_1 + R_3)$ from equation (2.35), that is $(31 + 64m)$, and dividing by 4 we get

$$559 + 4^5(2w + Q_{213}) + f(R_2, R_1, R_3,) + 16R_2 - 2 \times 91R_3 - 16m = 0 \quad (2.37)$$

$f(R_2, R_1, R_3)$ has a factor of 4, except for the last term which is

$$3 \times 2^5 R_2^{\,2} - 242R_3^{\,2} - 3(R_1^{\,2} + R_3^{\,2}).$$

Division of equation (2.37) by 2 gives a residual $(559 - 3(R_1 + R_3)^2)/2$.

Substitution in the class functions yields $(R_1 + R_3) = 4(r_i + r_j) + 3$, which when substituted into equation (2.37) divided by 2, gives a residual of $(265 + R_3(3R_1 - 121R_3 - 91))/2$.

Since R_1 and R_3 have opposite parity the second term is even so that the final residual is $265/2$ and no integer solutions are possible.

R_2 EVEN

The LHS of equation (2.31) is now odd so that K_{213} is even. Thus, for integer solutions of this equation

$$R_2 - (R_1 + R_3) = d + 2nw \qquad (2.38)$$

where $w = 0, 1, 2, 3\ldots$ and $d = 1, n_1, n_2, \ldots (2n-1), n_i \neq n$. Again, only one permissible d is found for a given n. For $n = 3, 7, 11, \ldots,$ (i.e. $(n+1)/2$ even, n in class $\bar{3}$) $d = (1+3n)/2$; whilst for $n = 5, 13, 17, \ldots, d = (n+1)/2$ (Table 2.7).

n	d	A	Coefficient
3	5	5	25
5	3	53	143
7	11	515	6363
11	17	43775	1616547
13	7	396533	9003383
17	9	32252273	2271909023

TABLE 2.7

Equations (2.34) and (2.35) still apply. R_2 will be in class $\bar{0}$ or $\bar{2}$. In both cases $(2R_2 + 1)$ falls in class $\bar{1}$, so that, if $g(R_1, R_3)$ is in class $\bar{3}$ $(n = 3, 7, 11\ldots)$ then $(1 + 2m)$ falls in class $\bar{3}$ so that m is odd. On the other hand, if $g(R_1, R_3)$ is in class $\bar{1}$ $(n = 5, 13, 17.)$ then $(1 + 2m)$ falls in class $\bar{1}$ and m is even.

2.3 Class Set $\bar{1}\,\bar{2}\,\bar{1}$

Here we have

$$(4R_1 + 1)^n = (4R_2 + 2)^n + (4R'_1 + 1)^n \qquad (2.39)$$

which, when expanded and divided by $4n$ gives

$$(4^{n-1}(R_1{}^n - R_2{}^n - R'_1{}^n) - A)/n \quad + \quad 4g(R_1, R_2, R'_1)$$
$$+ (R_1 - 2^{n-1}R_2 - R'_1) = 0 \qquad (2.40)$$

with

$$A = 2^{n-2} \qquad (2.41)$$

and

$$g(R_1, R_2, R'_1) = \sum_{j=0} 4^{n-3-j}\left(\frac{(n-1)j}{j!}\right)(R_1{}^{n-j-1} - 2^{j+1}R_2{}^{n-j-1} - R'_1{}^{n-j-1}) \qquad (2.42)$$

R_2 ODD

As for the above sets we can derive

$$R_1 - R_2 - R_1' = (1 + nK_{121})^{1/(n-1)}. \tag{2.43}$$

Since R_1 and R'_1 must have the same parity (equation 2.40), the LHS of equation (2.43) is odd, so that K_{121} must be even. With

$$R_1 - R_2 - R_1' = d + 2nw \tag{2.44}$$

then $d = 1, n_1, n_2, \ldots (2n-1)$ with $n_i \neq n$ and $w = 0, 1, 2, 3, \ldots$.

Equation (2.40) becomes:

$$(4^{n-1}d - A)/n + 4^{n-1}(2w + Q_{121}) + 4g(R_1, R_2, R_1') + (R_1 - 2^{n-1}R_2 - R_1') = 0 \tag{2.45}$$

Equation (2.39) may be put in the form:

$$(2R_2 + 1)^n = (R_1 - R'_1)h(R_1, R'_1)/2^{n-2} \tag{2.46}$$

where

$$h(R_1, R_1') = (x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots y^{n-1})$$

with $\quad x = (4R_1 + 1), y = (4R_1' + 1)$

This function will always be odd as it has n terms, all positive, except for the coefficient which equals n. R_1 and R_1' must have the same parity, so for integer solutions:

$$R_1 - R_1' = 2^{n-2}m = Am \tag{2.47}$$

m must be odd to conform to the parities of the other terms of equation (2.46). Values of A, d and the first term of equation (2.45) are listed in Table 2.8. By using the $\mathbb{Z}_4$ rules, the class structure of equation (2.46) can be deduced (Table 2.9).

n	d	A	Coefficient
3	5	2	$26 = 13A$
5	3	8	$152 = 19A$
7	11	32	$6432 = 201A$
11	17	512	$1620480 = 3165A$
13	7	2048	$9033728 = 4411A$

For $n = 3, 7, 11\ldots, d = (3n+1)/2; n$ in $\bar{1}, d = (n+1)/2$

TABLE 2.8

$2R_2+1$	m	$h(R_1, R_1')$	n	R_2
$\bar{3}$	$\bar{1}$	$\bar{3}$	$3, 7, 11\ldots$	odd
$\bar{1}$	$\bar{3}$	$\bar{3}$		even
$\bar{3}$	$\bar{3}$	$\bar{1}$	$5, 13\ldots$	odd
$\bar{1}$	$\bar{1}$	$\bar{1}$		even

TABLE 2.9

Again, these numerous restraints prevent any integer solutions. For example, when $n = 5$, equation (2.45) becomes

$$8 \times 19 + 4^4(2w + Q_{121}) + 4^3 g(R_1, R_2, R_1{'}) + (R_1 - R_1{'}) - 16R_2 = 0 \qquad (2.48)$$

With $(R_1 - R_1{'}) = 8m$ and class functions for m and R_2 from Table (2.9), division by 4^3 gives a residual of $5/2$, since the rows of the class functions are of the same parity.

R_2 EVEN

The left hand side of equation (2.43) will now be even so that K_{121} must be odd. This makes $d = 2, 4, 6, 8, \ldots 2(n-1)$. For integer solutions then, with $n = 3, 7, 11, \ldots$, $d = \frac{1}{2}(n+1)$; whilst for $n = 5, 13, 17, \ldots$, $d = (3n+1)/2$ (Table 2.10).

n	d	A	Coefficient
3	2	2	$10 = 5A$
5	8	8	$408 = 51A$
7	4	32	$2336 = 73A$
11	6	512	$571904 = 1117A$
13	20	2048	$25810944 = 12603A$

TABLE 2.10

The same arguments apply as for the odd R_2.

2.4 CLASS SET $\bar{1}\ \bar{0}\ \bar{1}$

For this case

$$(4R_1 + 1)^n = (4R_0)^n + (4R'_1 + 1)^n \qquad (2.49)$$

which, when expanded and divided by $4n$ gives:

$$4^{n-1}(R_1{}^n - R_0{}^n - R'_1{}^n)/n + 4f(R_1, R'_1) + (R_1 - R'_1) = 0 \qquad (2.50)$$

with $\quad f(R_1, R'_1) = \sum_{j=1}^{n-2} (n_j/n)4^{n-j-2}({R_1}^{n-j} - {R_1'}^{n-j})/j!$

with $\quad n_j = n(n-1)\ldots(n-j+1)$.

Unlike the other class sets, for $< \overline{1}\,\overline{0}\,\overline{1} >$ it is necessary that $(R_1 - R_0 - R'_1)$ is not prime to n. Hence, if integer solutions are to be obtained:

$$R_1 - R_0 - R'_1 = nq \tag{2.51}$$

where q is integer and has the same parity as R_0 because $(R_1 - R'_1)$ must be even (equation 2.50)

Equation (2.49) may be expressed as

$$R_0{}^n = (R_1 - R'_1)(h(R_1, R'_1))/4^{n-1} \tag{2.52}$$

$h(R_1, R'_1)$ is the same as for equation (2.46) and so is always odd.
For integer solutions,

$$R_1 - R'_1 = 4^{n-1}m \tag{2.53}$$

When R_0 is odd, m must also be odd and when R_0 is even m is even. The numerical term of $h(R_1, R'_1)$ equals n so that this function falls in class $\overline{3}$ for $n = 3, 7, 11, \ldots$, and in class $\overline{1}$ for $n = 5, 13, 17, \ldots$.

From equation (2.51), $(R_1 - R'_1) = R_0 + nq$, and with $h(R_1, R'_1) = 4R'_3 + 3 = 8R''_3 + 3$ (since R'_3 is even), and with $R_0{}^n = R_0 + nK_0$, equation (2.52) may be rewritten as:

$$(4^{n-1} - 3)R_0 - 3nq + 4^{n-1}nK_0 = 8R_0R''_3 + 8nqR''_3 \tag{2.54a}$$

which applies to n in $\overline{3}$. When n is in $\overline{1}$, the corresponding equation is

$$(4^{n-1} - 1)R_0 - nq + 4^{n-1}nK_0 = 4R_1{}'R_0 + 4nqR'_1. \tag{2.54b}$$

R_0EVEN

When R_0 is even q will be even and these quantities must be in the same class ($\overline{0}$ or $\overline{2}$) to satisfy the integer requirements of equations (2.54a) and (2.54b).

With $R_0 = 4r_0$, equation (2.52) becomes:

$$r_0{}^n = (m/4^n)h(R_1, R'_1) \tag{2.55}$$

If m is divisible by 4^n, then the parity of r_0 is considered. If r_0 is even, then $(m/4^n)$ must be divisable by 4^n or 2^n and so on. If r_0 is odd, we revert back to the case where R_0 is odd.

On the other hand, if $R_0 = 4R_2 + 2$, equation (2.52) becomes

$$(2r_2 + 1)^n = (m/2^n)h(R_1, R'_1). \tag{2.56}$$

Since the left hand side of this equation is odd, the solution reverts back to the R_0 odd case.

R_0 ODD

Here q will also be odd and R_0 and q must fall in the same class (either $\overline{1}$ or $\overline{3}$) in order to give an integer result for equation (2.54a) when n is in $\overline{3}$, and in opposite classes when n is in $\overline{1}$ in order to satisfy equation (2.54b).

With $({R_1}^n - {R_0}^n - {R_1}'^n) = R_1 - R_0 - {R_1}' + nQ_{101}$, and using equation (2.51), equation (2.50) becomes:

$$(4^{n-1} + n)q + R_0 + 4^{n-1}Q_{101} + 4f(R_1, {R_1}') = 0 \tag{2.57}$$

For the $(n+1)/2$ even series $(n \in \overline{3})$, let $R_0 = 4{r_3}' + 3$ and $q = 4{r_3}'' + 3$, so that equation (2.57) becomes

$$(3 \times 4^{n-1} + 3n + 3) + (4^n + 4n){r_3}'' + 4{r_3}' + 4^{n-1}Q_{101} + 4f(R_1, {R_1}') = 0 \tag{2.58}$$

As an example, take $n = 3$, then equation (2.58) is:

$$60 + 4(19{r_3}'' + {r_3}') + 16Q_{101} + 4(R_1 - {R_1}')(R_1 + R_1\prime) = 0. \tag{2.59}$$

From equations (2.51) and (2.53) ${r_3}''$ and ${r_3}'$ must have the same parity so that division of equation (2.59) by 8 gives a residual of $15/2$ so that there are no integer solutions. The same applies for higher n. When q and R_0 are in class $\overline{1}$, the residual is $5/2$.

3. METHOD 2

3.1 Modular Classes

Here we use Fermat's theorem

$$N^n = N + nK_N \tag{3.1}$$

in order to reduce powers appropriately. For example, with

$$(4R_i + i)^n = (4R_j + j)^n + (4R_k + k)^n. \tag{3.2}$$

We can (a) take N as the $\mathbb{Z}_4$ class function, $(4R_i + i)$, or (b) expand equation (3.2) via the binomial theorem and consider N as $(4R_i)^n, (4R_i)^{n-2}, (4R_i)^{n-4}$ or $(4R_j)^n \ldots$, taking one or more of these power terms (which all fall in class $\overline{0}$) according to the class-set and the characteristics of the relationships within. We shall first give tables setting out the features of the important variable K_N. These tables are of general interest and can be used (with extensions as required) for any analysis involving equation (3.1). Additional tables are given for the individual sets as required.

The K_N values have the form

$$K_{N_i} = Ag_{ij}$$

where A is a multiple of 4, except for class $\overline{2}$, where $A = 2.g$ is always odd, i represents the class of N and j the class of g. N can be characterised further by the class of the row in which g falls. Further, when a series of N fall in the same row class, R_k, say, these integers can be uniquely characterised via the K_{N_i} patterns (Tables 3.1 to 3.4).

Class $\overline{3}$ numbers are the most complex. When R_3 falls in class $\overline{3}$, no clear patterns for $R(g_{3_j})$ emerges. On the other hand, class $\overline{2}$ numbers are very simple in structure. Probably because N^n falls in class $\overline{0}$ so that nK_N must fall in class $\overline{2}$ and for all odd n, K_n falls in class $\overline{2}$, so cannot be a multiple of 4. $n = 3$ has been used for the tables but the results for higher powers are similar (see Table 3.5).

Class of R(g_{oi})	t_o		
	$g_{01} \in \overline{1}$	$g_{03} \in \overline{3}$	R_0 in class $\overline{0}$
$\overline{0}$	26	14	$R_0 = t_0 T + 32Tt$
$\overline{1}$	2	22	$K_{0i} = 8Tg_{0i}$
$\overline{2}$	10	30	$T = 2, 4, 8, 16, 32, 64\ldots$
$\overline{3}$	18	6	$t = 1, 2, 3, \ldots.$

TABLE 3.1a : Class $\overline{0}$

Class of $R(g_{0i})$	g_{0l} in $\overline{1}$			g_{03} in $\overline{3}$		
	t_0	$t_0{}'$	class of row of $R(g_{oi})$	t_0	t_0'	class of row of $R(g_{o3})$
$\overline{0}$	26	0	$\overline{1}$	14	0	$\overline{1}$
		1	$\overline{2}$		1	$\overline{2}$
		2	$\overline{3}$		2	$\overline{3}$
		3	$\overline{0}$		3	$\overline{0}$
$\overline{1}$	2	0	$\overline{1}$	22	0	$\overline{2}$
		1	$\overline{2}$		1	$\overline{3}$
		2	$\overline{3}$		2	$\overline{0}$
		3	$\overline{0}$		3	$\overline{1}$
$\overline{2}$	10	0	$\overline{2}$	30	0	$\overline{3}$
		1	$\overline{3}$		1	$\overline{0}$
		2	$\overline{0}$		2	$\overline{1}$
		3	$\overline{1}$		3	$\overline{2}$
$\overline{3}$	18	0	$\overline{3}$	6	0	$\overline{3}$
		1	$\overline{0}$		1	$\overline{0}$
		2	$\overline{1}$		2	$\overline{1}$
		3	$\overline{2}$		3	$\overline{2}$

R_0 in $\overline{2}$

$R_0 = t_0 + 32t$

$K_{0i} = 8g_{0i}$

$t = t_0{}' + 4t'$ with

$t' = 1, 2, 3, 4 \ldots$

TABLE 3.1b : Class $\overline{0}$

Class of $R(g_{0i})$	g_{0i} and R_0 in $\overline{1}$			g_{0i} and R_0 in $\overline{3}$		
	t_0	t	class pattern for row of $R(g_{0l})$	t_0	t	class pattern for row of $R(g_{03})$
$\overline{0}$	5	odd	$\overline{0}, \overline{1}, \overline{2}, \overline{3}$	7	even	$\overline{2}, \overline{3}, \overline{0}, \overline{1}$
$\overline{1}$	1	even	$\overline{0}, \overline{1}, \overline{2}, \overline{3}$	3	odd	$\overline{3}, \overline{0}, \overline{1}, \overline{2},$
$\overline{2}$	5	even	$\overline{1}, \overline{2}, \overline{3}, \overline{0}$	7	odd	$\overline{0}, \overline{1}, \overline{2}, \overline{3},$
$\overline{3}$	1	odd	$\overline{2}, \overline{3}, \overline{0}, \overline{1}$	3	even	$\overline{0}, \overline{1}, \overline{2}, \overline{3},$

$$R_0 = t_0 + 8t;$$

$$K_{0i} = 4g_{0i}$$

TABLE 3.1c : Class $\overline{0}$

Class of $R(g_{li})$	g_{li} and R_1 in $\overline{1}$			g_{li} and R_1 in $\overline{3}$		
	t_0	t	class pattern for row of $R(g_{11})$	t_0	t	class pattern for row of $R(g_{13})$
$\overline{0}$	5	even	$\overline{0}, \overline{2}, \overline{0}, \overline{2},$	11	even	$\overline{1}, \overline{3}, \overline{1}, \overline{3},$
		odd	$\overline{3}, \overline{1}, \overline{3}, \overline{1},$		odd	$\overline{0}, \overline{2}, \overline{0}, \overline{2},$
$\overline{1}$	1	even	$\overline{0}, \overline{2}, \overline{0}, \overline{2},$	7	even	$\overline{3}, \overline{1}, \overline{3}, \overline{1},$
		odd	$\overline{3}, \overline{1}, \overline{3}, \overline{1},$		odd	$\overline{2}, \overline{0}, \overline{2}, \overline{0},$
$\overline{2}$	13	even	$\overline{3}, \overline{1}, \overline{3}, \overline{1},$	3	even	$\overline{1}, \overline{3}, \overline{1}, \overline{3},$
		odd	$\overline{2}, \overline{0}, \overline{2}, \overline{0},$		odd	$\overline{0}, \overline{2}, \overline{0}, \overline{2},$
$\overline{3}$	9	even	$\overline{3}, \overline{1}, \overline{3}, \overline{1},$	15	even	$\overline{2}, \overline{0}, \overline{2}, \overline{0},$
		odd	$\overline{2}, \overline{0}, \overline{2}, \overline{0},$		odd	$\overline{1}, \overline{3}, \overline{1}, \overline{3},$

$$R_1 = t_0 + 16t;$$

$$K_{li} = 8g_{li}$$

TABLE 3.2a : Class $\overline{1}$

Class of $R(g_{li})$	g_{li} in $\overline{1}$			g_{li} in $\overline{3}$		
	t_0	t	class pattern for row of $R(g_{11})$	t_0	t	class pattern for row of $R(g_{13})$
$\overline{0}$	14	even	$\overline{1},\overline{3},\overline{1},\overline{3},$	26	even	$\overline{3},\overline{1},\overline{3},\overline{1},$
		odd	$\overline{0},\overline{2},\overline{0},\overline{2},$		odd	$\overline{2},\overline{0},\overline{2},\overline{0},$
$\overline{1}$	6	even	$\overline{0},\overline{2},\overline{0},\overline{2},$	18	even	$\overline{2},\overline{0},\overline{2},\overline{0},$
		odd	$\overline{3},\overline{1},\overline{3},\overline{1},$		odd	$\overline{1},\overline{3},\overline{1},\overline{3},$
$\overline{2}$	30	even	$\overline{2},\overline{0},\overline{2},\overline{0},$	10	even	$\overline{1},\overline{3},\overline{1},\overline{3},$
		odd	$\overline{1},\overline{3},\overline{1},\overline{3},$		odd	$\overline{0},\overline{2},\overline{0},\overline{2},$
$\overline{3}$	22	even	$\overline{1},\overline{3},\overline{1},\overline{3},$	2	even	$\overline{0},\overline{2},\overline{0},\overline{2},$
		odd	$\overline{0},\overline{2},\overline{0},\overline{2},$		odd	$\overline{3},\overline{1},\overline{3},\overline{1},$

R_1 in $\overline{2}$

$R_1 = t_0 + 32t$

$K_{li} = 16g_{li}$

TABLE 3.2b : Class $\overline{1}$

Class of $R(g_{li})$	t_0	
	g_{11} in $\overline{1}$	g_{13} in $\overline{3}$
$\overline{0}$	44	4
$\overline{1}$	28	52
$\overline{2}$	12	36
$\overline{3}$	60	20

R_1 in class $\overline{0}$

$R_1 = 4r_0, r_0$ odd

$R_1 = t_0 + 64t$

$t = 1, 2, 3, \ldots$

$K_{li} = 32g_{li}$

TABLE 3.2c : Class $\overline{1}$

Class of $R(g_{li})$	t_0	
	g_{11} in $\overline{1}$	g_{13} in $\overline{3}$
$\overline{0}$	3	9
$\overline{1}$	15	5
$\overline{2}$	11	1
$\overline{3}$	7	13

R_1 in class $\overline{0}$

$R_1 = 4r_0, r_0$ even

$R_1 = T + 2Tt$ (T series)

$T = 8, 16, 32, 64, \ldots$

$t = 1, 2, 3, \ldots$

$R_1 = t_0 T + 32Tt$

$(g_{li}, R(g_{li}))$ series

$K_{li} = 8Tg_{li}$

TABLE 3.2d : Class $\overline{1}$

Class of $R(g_{3i})$	t_0	
	g_{31} in $\overline{1}$	g_{33} in $\overline{3}$
$\overline{0}$	29	9
$\overline{1}$	21	1
$\overline{2}$	13	25
$\overline{3}$	5	17

R_3 in class $\overline{1}$

$R_3 = t_0 + 32t$

$K_{3i} = 16g_{3i}$

TABLE 3.3a : Class $\overline{3}$

Class of $R(g_{3i})$	g_{3i} in $\overline{1}$ and R_3 in $\overline{0}$			g_{3i} in $\overline{3}$ and R_3 in $\overline{2}$		
	t_0	t	class pattern for row of $R(g_{31})$	t_0	t	class pattern for row of $R(g_{33})$
$\overline{0}$	0	even	$\overline{0},\overline{2},\overline{0},\overline{2},$	6	even	$\overline{3},\overline{1},\overline{3},\overline{1},$
		odd	$\overline{3},\overline{1},\overline{3},\overline{1},$		odd	$\overline{2},\overline{0},\overline{2},\overline{0},$
$\overline{1}$	12	even	$\overline{1},\overline{3},\overline{1},\overline{3},$	2	even	$\overline{3},\overline{1},\overline{3},\overline{1},$
		odd	$\overline{0},\overline{2},\overline{0},\overline{2},$		odd	$\overline{2},\overline{0},\overline{2},\overline{0},$
$\overline{2}$	8	even	$\overline{3},\overline{1},\overline{3},\overline{1},$	14	even	$\overline{2},\overline{0},\overline{2},\overline{0},$
		odd	$\overline{2},\overline{0},\overline{2},\overline{0},$		odd	$\overline{1},\overline{3},\overline{1},\overline{3},$
$\overline{3}$	4	even	$\overline{1},\overline{3},\overline{1},\overline{3},$	10	even	$\overline{2},\overline{0},\overline{2},\overline{0},$
		odd	$\overline{0},\overline{2},\overline{0},\overline{2},$		odd	$\overline{1},\overline{3},\overline{1},\overline{3},$

$$R_3 = t_0 + 16t$$
$$K_{3i} = 8g_{3i}$$

TABLE 3.3b : Class $\overline{\mathbf{3}}$

g_{31} in class $\overline{1}$	$R_3 = (12T - 1) + 8Tt$	$T = 1, 2, 4, 8, 16, \ldots$
g_{33} in class $\overline{3}$	$R_3 = (4T - 1) + 8Tt$	$t = 1, 2, 3, \ldots$

R_3 in class $\overline{3}$

$$K_{3i} = 32Tg_{3i}$$

TABLE 3.3c : Class $\overline{\mathbf{3}}$

Class of $R(g_{2i})$	g_{21} in class $\overline{1}$		g_{23} in class $\overline{3}$	
	t_o	Class of R_2	t_o	Class of R_2
$\overline{0}$	8	$\overline{0}$	1	$\overline{1}$
$\overline{1}$	2	$\overline{2}$	3	$\overline{3}$
$\overline{2}$	4	$\overline{0}$	5	$\overline{1}$
$\overline{3}$	6	$\overline{2}$	7	$\overline{3}$

$(R_2 = t_0 + 8t;\ K_{2i} = 2g_{2i}.)$

TABLE 3.4 : Class $\overline{\mathbf{2}}$

3.2. CLASS SET $\overline{0}\ \overline{1}\ \overline{3}$

The triple becomes

$$(4R_0)^n = (4R_1 + 1)^n + (4R_3 + 3)^n. \tag{3.3}$$

We shall consider $n = 3$ and $n = 5$, as each of these exponents represents class $\overline{3}$ and class $\overline{1}$, respectively, with specific functions.

$n = 3$

It is convenient to use the $(4R_i)^n$ form of breakdown in this case, with $i = 1$ and 3. that is,

$$(4R_1)^3 = 4R_1 + 3K_{0j} \tag{3.4}$$

and

$$(4R_3)^3 = 4R_3 + 3K_{0k}. \tag{3.5}$$

The subscripts on K indicate that $\overline{0}$ class is used for the Fermat reduction and that g falls in class j for R_1 and g falls in class k for R_3 (j and k are either class $\overline{1}$ or $\overline{3}$ since g is always odd). Taking R_1 odd and R_3 even gives

$$K_{0j} = 4g_{0j} \tag{3.5}$$

$$K_{0k} = 8Tg_{0k}$$

where $T = 1, 2, 4, 8, 16, \ldots,$ (Table 3.5).

Expanding the components b and a from equation (3.3) we get

$$(4R_0)^3 = 3 \times 16({R_1}^2 + 3{R_3}^2) + 16R_1 + 28 \times 4R_3 + 3 \times 4g_{0j} + 3 \times 8Tg_{ok} + 28 \tag{3.6}$$

If $j = 3$, $g_{0j} = 4\overline{r}_3 + 3$, where the bar on r indicates a row for g.
Division of equation (3.6) by 16 gives

$$4(R_0)^3 = 3({R_1}^2 + 3{R_3}^2) + R_1 + 7R_3 + 3\overline{r}_3 + 3(Tg_{0k})/2 + 4 \tag{3.7}$$

R_1 must be in class $\overline{3}$ to conform to g_{03} (Table 3.5). Hence R_3 must be in class $\overline{0}$ (equations 2.10, 2.17) and $T \geq 2$ (Table 3.1a). With $T = 2$, $\overline{r}_3$ must be odd (equation 3.7). With the class structure $(R_1, R_3) = \ < \overline{3}, \overline{0} >$, equations (2.10,2.17) and taking into account the class structure of $\overline{r}_3$ (Table 3.1c) and g_{0k} (Table 3.1a), division of equation (3.7) by 4 gives a residual of 23/2. When $T > 2, \overline{r}_3$ is even and a similar analysis gives a non-integer result.

If $j = 1, g_{0j} = 4\bar{r} + 1$ and $T = 1$. R_1 is now in class $\bar{1}$, and R_3 in class $\bar{2}$. Again, a non-integer result applies (Tables 3.1a and 3.1c).

$n = 5$

Following the same method as for $n = 3$, we get

$$(4R_0)^5 = 5 \times 4^4({R_1}^4 + 3{R_3}^4) + 10 \times 4^3({R_1}^3 + 9{R_3}^3) + 10 \times 16({R_1}^2 + 27{R_3}^2)$$
$$+ 6 \times 4R_1 + 406 \times 4R_3 + 5 \times 8Tg_{0k} + 5 \times 4g_{0j} + 244. \quad (3.8)$$

With $g_{03} = 4\bar{r}_3 + 3$, substituted into equation (3.8) and dividing by 16, we obtain

$$4^3{R_0}^5 = 5 \times 4^2({R_1}^4 + 3{R_3}^4) + 10 \times 4({R_1}^3 + 9{R_3}^3) + 10({R_1}^2 + 27{R_3}^2)$$
$$+ (3R_1/2) + (203R_3/2) + 5Tg_{0k}/2 + 5\bar{r}_3 + 19. \quad (3.9)$$

With g_{0j} in $\bar{3}$, R_1 is in $\bar{1}$ (Table 3.5) so that R_3 must be in $\bar{2}$ (equations 2.10,2.17) and $T = 1$ (Table 3.1b). Consideration of the last five terms of equation (3.9), taking into account the compatibility of classes for R_1, R_3, g_{0k} and $\bar{r}_3$, shows that residuals of $B/2$ ($B = 225, 235$, or 245) occur so that there are no integer solutions. A similar analysis with $g_{0j} = 4\bar{r}_1 + 1$ gives the same result.

In the same manner, non-integer results apply when R_1 is even and R_3 odd, and for the higher n values.

n	Ri	$((4R_i)^n - 4R_i)/n = K_{0i}$	Class for g_{0i}	R_i equations
3	odd	$4g_{0i}$	$\bar{3}$	$3 + 4t$
			$\bar{1}$	$1 + 4t$
5	odd	$4g_{0i}$	$\bar{3}$	$1 + 4t$
			$\bar{1}$	$3 + 4t$
3	even	$8Tg_{0i}$	$\bar{3}$	$6T + 8Tq$
		$T = 1, 2, 4, 8, 16\ldots$	$\bar{1}$	$2T + 8Tq$
5	even	$8Tg_{0i}$	$\bar{3}$	$2T + 8Tq$
		$T = 1, 2, 4, 8, 16\ldots$	1	$6T + 8Tq$

TABLE 3.5

3.3 CLASS SET $\bar{2}\ \bar{1}\ \bar{3}$

In this case the triple is

$$2^n(2R_2+1)^n = (4R_1+1)^n + (4R_3+3)^n \qquad (3.10)$$

which obviously has a similar form to equation (3.3) in that the left hand side is divisible by 16 when $n > 3$. Consequently, for $n > 3$ the same method applies as for the class set $\bar{0}\,\bar{1}\,\bar{3}$.

When $n = 3$, $(R_1 + R_3 + 1)$ is in class $\bar{2}$ so that g_{0i} (equation 3.6) equals $(4\bar{r}_1 + 1)$ when $T > 1$, but it equals $(4\bar{r}_3 + 3)$ when $T = 1$, which is the opposite situation from the class set $\bar{0}\,\bar{1}\,\bar{3}$. However, because $(2R_2 + 1)$ is odd, a residual of $(2R_2 + 1)/2$ is obtained, so that there are no integer solutions for $n = 3$ as well.

3.4 CLASS SET $\bar{1}\ \bar{2}\ \bar{1}$

For this set

$$(4R_1+1)^n - (4R_1{}'+1)^n = (4R_2+2)^n \qquad (3.11)$$

Expanding equation (3.11) by the binomial theorem and using Fermat's theorem to simplify the $(4R_i)^n$ terms to $(4R_i + nK_{ij})$, gives for $n = 3$

$$3\times 16(R_1{}^2 - R_1{}'^2) + 16(R_1 - R_1{}') + 3(K_{0j} - K_{0j}{}') = 52R_2 + 3K_{0k}$$
$$+ 6\times 16R_2{}^2 + 8. \qquad (3.12)$$

Since $(R_1 - R_1{}') = 2^{n-2}m$, where m is odd (Table 2.9), for $n = 3$, $(R_1 - R_1{}') = 2m$ and so $(R_1 - R_1{}')$ is in class $\bar{2}$, but for $n > 3$, $(R_1 - R_1{}')$ is in class $\bar{0}$. Thus, for $n = 3$, R_1 and $R_1{}'$ are in opposite classes within the same parity group. That is, $R_1 = \bar{1}$ and $R_1{}' = \bar{3}$ or vice versa. Whilst even parity gives $R_1 = \bar{0}$ and $R_1{}' = \bar{2}$ or vice versa. This situation is important when considering the characteristics of K_{ij} (Tables 3.1 to 3.4).

Obviously, for $n > 3$ with R_1 and $R_1{}'$ in the same class, the situation is simpler.

$n = 3$, R_2 ODD

From Table 3.1c,

$$3(K_{0j} - K_{0j}{}') = 3\times 4(g_{0j} - g_{0j}{}') = 12\times 4(\overline{R}_3 - \overline{R}_1) + 24, \text{ and}$$

$3K_{0k} = 12g_{0k}$, the g values always being odd.

Using the class characteristics of the various R and g quantities (Table 3.1c) and with R_2 in class $\bar{1}$ or $\bar{3}$, it can be shown that equation (3.12) has no integer solutions.

$n = 3$, R_2 EVEN

In this case the left hand side of equation (3.12) remains the same, namely

$$3 \times 16(R_1^2 - {R_1'}^2) + 16(R_1 - R_1') + 12 \times 4(\overline{R}_3 - \overline{R}_1) + 24 \qquad (3.13)$$

whilst the right hand side becomes:

$$52R_2 + 24Tg_{0k} + 6 \times 16{R_2}^2 + 8 \qquad (3.14)$$

When $R_2 \in \overline{2}$, $T = 1$ (Table 3.1b) and when $R_2 \in \overline{0}$, $T = 2, 4, 8, ...$ (Table 3.1a). The g_{0k} term can be in class $\overline{1}$ or $\overline{3}$, with the respective rows giving the functions for R_2 in Tables 3.1a and 3.1b. For example, with R_2 in class $\overline{2}$, g_{0k} in class $\overline{1}$ and in a row in class $\overline{0}$, the term (3.14) becomes

$$52(26 + 32t) + 24(4(4r_0) + 1) + 6 \times 16{R_2}^2 + 8. \qquad (3.15)$$

Dividing the left hand side and right hand side by 32 gives a residual of 85/2. If g_{0k} is in class $\overline{3}$ in a row in class $\overline{0}$, a residual of 49/2 is obtained. Whereas if the rows are in class $\overline{1}$, respectively, residuals of 13/2 and 81/2 are obtained and so on.

$n = 5$, R_2 **ODD**

The left hand side of equation (3.11) may be expressed as

$$4R_1 + 1 - (4R_1' + 1) + 5(k_1 - k_1') \qquad (3.16)$$

where the class k values, k_i, are used in contrast to the $(4R_i)$ values, K_{0j}.
From Table 3.2a and equation (2.47), with R_1 and R_1' both odd, the term (3.16) becomes

$$32m + 5 \times 8 \times 4(r_i - r_i') \qquad (3.17)$$

where r_i refers to the row in class i occupied by the g values; R_1 and R_1' being in the same classes when $n > 3$.

When R_1 and R_1' are both even, the coefficient of the second term of (3.17) increases (Tables 3.2b,c and d) so that division by 32 still gives integer terms on the left hand side.

On the other hand, we use the binomial form for the right hand side of equation (3.11) and take the cubic and fifth power terms as Fermat reductions.

$$(4R_2)^5 = 4R_2 + 5K_{0k} \qquad (3.18)$$

$$40(4R_2)^3 = 40(4R_2 + 3K_{0k}). \qquad (3.19)$$

When R_2 is odd, from Table 3.1c

$$5K_{0k} = 20g_{0k} \text{ and } 120K_{0k} = 120 \times 4g_{0k}.$$

There are two classes R_2 can be in, t here representing the row.
(a) $R_2 = 1 + 4t$ and $g_{03} = 4\overline{R}_3 + 3, g_{01} = 4\overline{R}_1 + 1$.

The RHS of equation (3.11) becomes
$344+16 \times 41t + 5 \times 16\overline{R}_3 + 30 \times 16\overline{R}_1 + 5 \times 16 \times 32R_2{}^4 + 10 \times 16 \times 8R_2{}^2 + 5 \times 16 \times 4R_2 + 32.$ (3.20)

Division of the left hand side and right hand side by 16 gives the residual 43/2, so that there are no integer solutions.

(b) $R_2 = 3 + 4t$ and $g_{01} = 4\overline{R}_1 + 1, g_{03} = 4\overline{R}_3 + 3$ and the RHS becomes
$872 + 16 \times 41t + 5 \times 16\overline{R}_1 + 5 \times 16 \times 32R_2{}^4 + 10 \times 16 \times 8R_2{}^2 + 30 \times 16\overline{R}_3 + 5 \times 16 \times 4R_2 + 32$ (3.21)

Division by 16 gives the residual 109/2, so that there are no integer solutions for this case either. The same sort of analysis can be made when R_2 is even.

A method which is independent of the parity of R_2 is as follows, taking $n = 5$ as an example.
The left hand side of equation (3.11) is
$(4R_1)^5 - (4R_1')^5 + 20(R_1 - R_1') + 5 \times 4^4(R_1{}^4 - R_1'^4) + 10 \times 4^3(R_1{}^3 - R_1'^3) + 10 \times 16(R_1{}^2 - R_1'^2)$ (3.22)

The first three terms may be expressed

$$(4R_1)^5 - (4R_1')^5 + 20(R_1 - R_1') = 24(R_1 - R_1') + 20(g_{0j} - g_{0j}'), \quad (3.23)$$

Since R_1 and R_1' are in the same class $g_{0j} - g_{0j}' = 4(r_i - r_i')$ and from equation (2.47) $(R_1 - R_1') = 8m$. Thus, if r_i and r_i' are of opposite parity, division by 32 gives a residual of $(5/2)(r_i - r_i')$ on the LHS. Since the RHS may be expressed $32(2\mathrm{R}_2 + 1)^5$ division by 32 gives an integer result for the RHS, thus there are no integer solutions for the equation as a whole. On the other hand, if r_i and r_i' are of the same class, division of the LHS by 64 gives no fractional residual, whereas the RHS has a residual, in this case, of an odd number divided by 2 so there are no integer solutions.

3.5 CLASS SET $\overline{1}\ \overline{0}\ \overline{1}$

The relationships for this set are

$$(4R_1+1)^n - (4R_1{}'+1)^n = (4R_0)^n. \qquad (3.24)$$

Consider $n=3$. The expanded functions are

$$(4R_1)^3 - (4R_1{}')^3 + 3\times 16(R_1{}^2 - R_1{}'^2) + 12(R_1 - R_1{}') = (4R_0)^3. \qquad (3.25)$$

Fermat's reduction gives, when R_1 and $R_1{}'$ are both odd:

$$(4R_1)^3 - (4R_1{}')^3 = 4(R_1 - R_1{}') + 4(g_{0j} - g_{0i}{}') \qquad (3.26)$$

and since

$$(R_1 - R_1{}') = 4^{n-1}m \qquad (3.27)$$

R_1 and $R_1{}'$ will be in the same class. From Table 3.5

$$(g_{0i} - g_{0i}{}') = 4(r_i - r_i{}') \qquad (3.28)$$

where i is 1 or 3 according to the class g falls in. Thus

$$3\times 16(R_1{}^2 - R_1{}'^2) + 16(R_1 - R_1{}') + 16(r_i - r_i{}') = 4^3 R_0{}^3 \qquad (3.29)$$

If $(r_i - r_i{}')$ is odd, that is, r_i and $r_i{}'$ have opposite parity, division by 32 gives a residual of $(r_i - r_i{}')/2$.

When R_0 is odd, since $(R_1 - R_1') = 16m$ and m has the same parity as R_0

$$(R_1 - R_1') = 16\times 4R_j + A \qquad (3.30)$$

where j is the class for R and $A = 16$ or 48 with $j = 1$ or 3.

Table 3.1c shows that if g_{01} is in class $\overline{1}$ and has a row $R(g_{01})$ in class $\overline{0}$, then since t is odd, the row of $R(g_{01})$ will fall in class $\overline{0}$ when $t = 1+8w$, in class $\overline{1}$ when $t = 3+8w$, in class $\overline{2}$ when $t = 5+8w$ and in class $\overline{3}$ when $t = 7+8w$. If R_1 and R_1' have the row of $R(g_{01})$ in different classes, this would not be compatible with equation (3.30) so that

$$(r_i - r_i') = 4(\overline{r}_i' - \overline{r}_i') = 16(\overline{\overline{r}}_i - \overline{\overline{r}}_i') = \ldots \qquad (3.31)$$

where the succeeding r's represent the rows of the rows and so on, as in an indexed matrix [2].

When equation (3.29) is divided by 4^4 a residual of $R_0{}^3/4$ is obtained so that there are no integer solutions. The same result follows when $R(g_{01})$ falls in the other classes listed in Table 3.1c. If, on the other hand, R_0 is even, and $R_0 = 4r_2 + 2$, then

$$(4R_0)^3 = 2 \times 4^4 + 4^5(4{r_2}^3 + 6{r_2}^2 + 3r_2) \quad (3.32)$$

which when substituted into equation (3.29) and with m in class $\overline{0}$, division by 4^5 gives a residual of 1/2. But, if m is in class $\overline{2}$, division by 64 gives a residual of 1/2.

When R_0 is in class $\overline{0}$ and m in class $\overline{2}$ a residual of 1/2 is obtained as before. However, when m is in class $\overline{0}$, the analysis becomes more ambiguous. Nevertheless, we can use equation (2.55) to get

$$m = 4^3{r_0}^3/h(R_1, R'_1) \quad (3.33)$$

where $R_0 = 4r_0$ and $h(R_1, R'_1)$ is odd. Equation (3.29) may now be expressed

$$4^7{r_0}^3(3(R_1 + R'_1) + 1)/h(R_1, R'_1) + 4^7((\overline{\overline{\overline{r}}}_1 - \overline{\overline{\overline{r}}}_1\prime) = 4^6{r_0}^3 \quad (3.34)$$

Since the RHS will become non-integer first, on dividing by powers of 4, there are no integer solutions.

Similar analyses can be made for the higher powers and the different parities of the R's.

4. FINAL COMMENTS

Previously [3] we have used a modular ring $\mathbb{Z}_6$ and the Pythagorean $z - j$ grid to analyse even exponent triples and the emphasis there was on the structure of the ring and grid, confining the integers in well ordered categories. As noted above, the emphasis in the present paper is similar, being on the character of the integers themselves, their parity and class structure and the specific Fermat power reduction patterns. The two papers, [3] and this one, represent a new approach to a very old subject.

Since 2 is a prime, Method 2 can be applied to Pythagorean triples. In this case

$$N^2 = N + 2K_N. \quad (4.1)$$

The K_N and R functions are similar to those for the odd primes (Table 4.1) but are compatible enough to yield solutions.

Class of R_j	$K_{0i} = ((4R_j)^2 - (4R_j))/2$	R_j function	Class of g_{0i}
$\overline{0}$	$8Tg_{0i}$	$4T + 16Tt$	$\overline{3}$
		$12T + 16Tt$	$\overline{1}$
$\overline{2}$	$4g_{0i}$	$2 + 8t(r_2\text{even})$	$\overline{3}$
		$6 + 8t(r_2 \text{ odd})$	$\overline{1}$
$\overline{1}$	$2g_{0i}$	$1 + 4t$	$\overline{3}$
$\overline{3}$	$2g_{0i}$	$3 + 4t$	$\overline{1}$

$T = 1\,(r_0\text{odd})$
$T = 2, 4, 8, 16\ldots$
$(r_0\,\text{even})$
$t = 0, 1, 2, 3\ldots$

TABLE 4.1

Within $\mathbb{Z}_4$ the Pythagorean triples only fall in the Class sets $< \overline{1}\,0\,\overline{1} >$, $< \overline{1}\,0\,\overline{3} >$, when $(c - b)$ is odd; and $< \overline{1}\,\overline{1}\,\overline{0} >$, $< \overline{1}\,\overline{3}\,\overline{0} >$ when $(c - b)$ is even [4].
Consider the set $< \overline{1}\,\overline{0}\,\overline{3} >$ for which

$$(4R_1 + 1)^2 = (4R_0)^2 + (4R_3 + 3)^2 \tag{4.2}$$

If R_1 and R_0 are in class $\overline{2}$ and R_3 in class $\overline{1}$, say

$$\left\{ \begin{array}{l} (4R_1)^2 = 4R_1 + 2 \times 4g_{0i} \\ (4R_0)^2 = 4R_0 + 2 \times 4g_{0j} \\ (4R_3)^2 = 4R_3 + 2 \times 2g_{0k} \end{array} \right\} \tag{4.3}$$

so that equation (4.2) becomes

$$3R_1 - R_0 - 7R_3 = 2(g_{0j} - g_{0i}) + g_{0k} + 2. \tag{4.4}$$

With $R_1 = 4r_2 + 2, R_0 = 4r_2' + 2$, let r_2 be even and r_2' be odd, then $R_1 = (2 + 8t)$ and $R_0 = (6 + 8t')$, with the corresponding g values falling in class $\overline{3}\,(4r_3' + 3)$ and $\overline{1}$ $(4r_1 + 1)$, respectively. Since in class $\overline{1}$, $R_3 = (1 + 4t'')$ and the corresponding g falls in $\overline{3}\,(4r_3 + 3)$, Table 4.1. Substituting these values into equation (4.4) and dividing by 4 we find

$$6t - 2t' - 7t'' = 2r_1 - 2r_3' + r_3 + 2 \tag{4.5}$$

for which solutions are obviously possible. One such is the triple $< 169, 120, 119 >$ which has $t = 5$, $t' = 3$, $t'' = 7$ and $r_1 = 446$, $r_3' = 876$ and $r_3 = 833$.

For the $<\bar{1}\,0\,\bar{1}>$ case the rows of the components, R_1, R_0, and R_1' are either all odd or all even. This can be shown by taking two rows even (say R_1, R_0 in class $\bar{2}$) and one odd, say R_1' in class $\bar{1}$ so that, with

$$\left\{\begin{array}{l} (4R_1)^2 = 4R_1 + 2(4g_{0i}) \\ (4R_0)^2 = 4R_0 + 2(4g_{0j}) \\ (4R_1')^2 = 4R_1' + 2(2g_{0k}) \end{array}\right\} \tag{4.6}$$

and since

$$(4R_1 + 1)^2 = (4R_0)^2 + (4R_1' + 1)^2. \tag{4.7}$$

Combining equations (4.6) and (4.7) we find

$$3(R_1 - R_1') - R_0 = 2(g_{0j} - g_{0i}) + g_{0k}. \tag{4.8}$$

Substituting in the appropriate functions from Table 4.1 we obtain

$$6t - 3t'' - 2t' = \tfrac{1}{2} + 2(r_1 - r_3') + r_3 \tag{4.9}$$

which shows that this parity set is invalid for integer solutions. On the other hand, taking all rows odd, we get

$$3(R_1 - R_1') - R_0 = g_{0j} + g_{0k} - g_{0i}. \tag{4.10}$$

If we take R_1 and R_0 in class $\bar{3}$ and R_1' in class $\bar{1}$ then, from Table 4.1, g_{0i} and g_{0j} are in class $\bar{1}$ (with rows r_1, r_1') and $R_1 = (3+4t)$ and $R_0 = (3+4t')$, whilst $R_1' = (1+4t'')$ and g_{ok} is in class $\bar{3}$ (row r_3). Thus equation (4.10) becomes

$$3(t - t'') - t' = r_1' + r_3 - r_1. \tag{4.11}$$

Since the t and r values are not restricted, there can be solutions. For example, the triple $< 13, 12, 5 >$ fits the requirements with t, t'' and t' all equal to zero and r_3 and r_1 both equal to 8 and r_1' is zero. Another solution is the triple $< 205, 156, 133 >$ with $t = 12, t' = 9$ and $t'' = 8$. $r_1 = 2588$, $r_1' = 1511$ and $r_3 = 1080$.

The above methods are alternatives to the method used recently for odd exponent triples where the $\mathbb{Z}_6$ modular ring was employed [7,8].

Of course, equation (1.1) in Fermat terms is:

$$c^n = (b + a) + n(K_b + K_a) \tag{4.12}$$

which suggests that $c = (b + a)$. But that solution gives no integer solutions for equation (1.1). We need to show that $(c - (b + a))$ is prime to n or that $K_c = (K_b + K_a)$, which is basically what we have illustrated here, and this confirms the incompatibility of equations (1.1) and (4.12). Perhaps this is where Fermat reached when he made his fascinating claim!

"*Cubem autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duas ejusdem nominis fas est dividere: cujas rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caparet.*" *(On the other hand, it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or in general, any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which however the margin is not large enough to contain.)*

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