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UNIT COEFFICIENT SUMS FOR CERTAIN MORGAN–VOYCE NUMBERS

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1. BEGINNING

Studying the unique minimal and maximal integer Zeckendorf representations by Pell numbers [2], [3], [4], [5] led to the consideration [6, eqn. (2.7)] of those numbers which are common to both representations, namely, the *MinMax numbers.* This idea was then carried over to Jacobsthal numbers [7, eqn. (3.1)]. (Earlier, minimal and maximal representations by Fibonacci and Lucas numbers had been investigated in [1].)

Here, the corresponding situation existing for Morgan–Voyce numbers is to be disclosed. Though the results are perhaps not quite so elegant as those for Pell numbers, they are nevertheless of intrinsic interest and value.

Morgan–Voyce polynomials $B_n(x)$ are defined [9] (with slight modification here) by the recursion

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \tag{1.1}$$

with

$$B_0(x) = 0, \quad B_1(x) = 1. \tag{1.2}$$

Integers resulting when we substitute $x = 1$ in (1.1) and (1.2) may sensibly be referred to as the *Morgan–Voyce numbers* $B_n(1) \equiv B_n$.

Our primary aim is to examine the properties of those integers, designated by $\mathcal{B}_n$, for which

$$\mathcal{B}_n = \sum_{i=1}^{n} B_i \tag{1.3}$$

$$= F_{2n+1} - 1 \quad \text{by } [8; (2.2), (2.22), (4.2), x = 1], \tag{1.4}$$

and related matters. Coefficients in the summation (1.3) are obviously all unity. This special summation (1.3) corresponds to the genesis of the MinMax representation numbers in the Pell context [6, eqn. (2.7)].

One doesn't have to be psychic to realise that, with $\mathcal{B}_0$ taken as 0, the sequence materializes as:

(1.5)

$\mathcal{B}_0$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$	$\mathcal{B}_7$	$\mathcal{B}_8$	$\mathcal{B}_9$	$\mathcal{B}_{10}$	$\cdots$
0	1	4	12	33	88	232	609	1596	4180	10945	$\cdots$

i.e., an odd number followed by two even numbers, as is inevitable from (1.4).

Extending the subscripts through negative integers, we see, mirrored in a glass clearly, that

(1.6) $$\mathcal{B}_{-n} = \mathcal{B}_{n-1},$$

e.g., $\mathcal{B}_{-1} = 0$.

Because of the coverage of the content in [2]–[6] for Pell numbers, we may be excused for providing a somewhat abbreviated account here in the Morgan–Voyce case. Only a sample of the possibilities which give a feeling for the material is herewith displayed.

2. MIDDLE: $\mathcal{B}_n$

A. Properties of $\mathcal{B}_n$

Elementary detective procedures reveal the following information about $\{\mathcal{B}_n\}$.

(2.1) *Recurrence:* $$\mathcal{B}_n = 3\mathcal{B}_{n-1} - \mathcal{B}_{n-2} + 1.$$

(2.2) *Generating function:* $$\sum_{i=1}^{\infty} \mathcal{B}_i x^{i-1} = \left[1 - (4x - 4x^2 + x^3)\right]^{-1}.$$

(2.3) *Binet form:* $$\mathcal{B}_n = \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} - 1 \quad \text{by (1.4)},$$

where

(2.4) $$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

(2.5) *Simson formula:* $$\mathcal{B}_{n+1}\mathcal{B}_{n-1} - \mathcal{B}_n^2 = -\mathcal{B}_n.$$

(2.6) *Summations:* $$\sum_{i=1}^{n} \mathcal{B}_{2i} = F_{2n}F_{2n+3} - n,$$

(2.7) $$\sum_{i=1}^{n} \mathcal{B}_{2i-1} = F_{2n}F_{2n+1} - n,$$

(2.8) $$\sum_{i=1}^{n} \mathcal{B}_i = F_{2n+2} - (n+1),$$

(2.9) $$\sum_{i=1}^{n} (-1)^{i+1}\mathcal{B}_i = (-1)^{n+1}\left\{F_{n+1}^2 + \frac{1+(-1)^n}{2}\right\}.$$

Proof of (2.9): Subtract (2.6) from (2.7). Then, by the Simson formula for Fibonacci numbers, (α). $$\mathcal{B}_1 - \mathcal{B}_2 + \mathcal{B}_3 - \mathcal{B}_4 + \cdots - \mathcal{B}_{2n} = -F_{2n}F_{2n+2} = -F_{2n+1}^2 + 1.$$

Add $\mathcal{B}_{2n+1}$ to both sides to get

(β). $$\mathcal{B}_1 - \mathcal{B}_2 + \mathcal{B}_3 - \cdots + \mathcal{B}_{2n+1} = F_{2n+2}^2 \quad \text{by (1.4), (2.3), (2.4).}$$

Combining (α) and (β) yields (2.9).

(2.10) *Other simple properties:* $\mathcal{B}_n - \mathcal{B}_{n-1} = F_{2n} = F_nL_n,$

(2.11) $\mathcal{B}_n + \mathcal{B}_{n+1} = L_{2n} - 2(= L_n^2, n \text{ odd}),$

(2.12) $\mathcal{B}_n - \mathcal{B}_{n-2} = L_{2n-1},$

(2.13) $\mathcal{B}_n^2 - \mathcal{B}_{n-1}^2 = F_{4n} - 2F_{2n},$

(2.14) $\mathcal{B}_n^2 - \mathcal{B}_{n-2}^2 = 3F_{4n-2}.$

Note the factors in (2.10), (2.13), and (2.14). E.g., in (2.10), $\mathcal{B}_{10} - \mathcal{B}_9 = 6,765(= F_{20}) = 55 \times 123(= F_{10}L_{10})$.

Divertissement

Temporarily writing

(2.1)′ $$\mathcal{B}'_n = \mathcal{B}_{n+1} + \mathcal{B}_{n-1} = 3\mathcal{B}_n + 1 \quad \text{by (2.1),}$$

and it is natural to consider this sum as an entity *per se*, so that $\mathcal{B}'_0 = 1, \mathcal{B}'_1 = 4, \mathcal{B}'_2 = 13, \mathcal{B}'_3 = 37, \mathcal{B}'_4 = 100, \cdots$, we readily arrive *inter alia* at

(2.15) $\mathcal{B}'_n = 3\mathcal{B}'_{n-1} - \mathcal{B}'_{n-2} + 2$ (*recurrence*),

(2.16) $\mathcal{B}'_n - \mathcal{B}'_{n-1} = 3F_{2n},$

(2.17) $\mathcal{B}'_n + \mathcal{B}'_{n-1} = 3L_{2n} - 4,$

(2.18) $\mathcal{B}'_n - \mathcal{B}'_{n-2} = 3(\mathcal{B}_n - \mathcal{B}_{n-2}) = 3L_{2n-1}$ by (2.12),

(2.19) $\sum_{i=1}^{n} \mathcal{B}'_i = 3\sum_{i=1}^{n} \mathcal{B}_i + n,$

(2.20) $\mathcal{B}'_{-n} = \mathcal{B}'_{n-1}.$

Other summations corresponding to (2.6), (2.7), (2.9) are left to the reader, with the knowledge that the analogue of (2.9) is somewhat ungainly. Obvious additional analogues which may be adduced are those corresponding to (2.2), (2.3), (2.5).

B. The $\mathcal{B}_n^{(m)}$ Grid

Interestingly, if we continue with the superscript dash (′) symbolism to establish $\mathcal{B}''_n = \mathcal{B}'_{n+1} + \mathcal{B}'_{n-1}, \mathcal{B}'''_n = \mathcal{B}''_{n+1} + \mathcal{B}''_{n-1}, \cdots$,

(2.21) $$\mathcal{B}_n^{(m+1)} = \mathcal{B}_{n+1}^{(m)} + \mathcal{B}_{n-1}^{(m)} \qquad (\mathcal{B}_n^{(0)} = \mathcal{B}_n),$$

then we may verify that, for instance,

(2.22) $$\mathcal{B}_{-n}^{(m)} = \mathcal{B}_{n-1}^{(m)},$$

(2.23) $$\mathcal{B}_0^{(m)} = 3^m - 2^m \qquad (induction),$$

(2.24) $$\mathcal{B}_n^{(m)} - \mathcal{B}_{n-1}^{(m)} = 3^m F_{2n},$$

(2.25) $$\mathcal{B}_n^{(m)} + \mathcal{B}_{n-1}^{(m)} = 3^m L_{2n} - 2^{m+1},$$

(2.26) $$\mathcal{B}_n^{(m)} - \mathcal{B}_{n-2}^{(m)} = 3^m L_{2n-1},$$

(2.27) $$\mathcal{B}_n^{(m)} = 3^m \mathcal{B}_n + \mathcal{B}_0^{(m)} = 3^m F_{2n+1} - 2^m,$$

(2.28) $$\sum_{i=1}^{n} \mathcal{B}_i^{(m)} = 3^m \sum_{i=1}^{n} \mathcal{B}_i + n\mathcal{B}(_0^{(m)}.$$

Thus, e.g., $\mathcal{B}_2^{(\mathrm{iv})} = 389$.

Suppose we construct the *representation–derived grid* for $\mathcal{B}_n^{(m)}$ as both m (columns) and n (rows) vary integrally and infinitely ($m = 0, 1, \cdots; n = \cdots, -2, -1, 0, 1, 2 \cdots$). Some intriguing patterns then appear. For instance, we find that for

(2.29) rows(n) : $$\mathcal{B}_n^{(m)} = 3\mathcal{B}_n^{(m-1)} + 2^{m-1} \ (m \geq 1),$$

(2.30) columns(m) : $$\mathcal{B}_n^{(m)} = 3\mathcal{B}_{n-1}^{(m)} - \mathcal{B}_{n-2}^{(m)} + 2^m \ (recurrence).$$

Fuller investigations of the potential of this numerical grid, and of the properties of the generalized systems $\mathcal{B}_n^{(m)}$, are not pursued here. Earlier formulas in (A) are re–inforced when $m = 0$ in (B).

3. MIDDLE: $\mathcal{C}_n$.

A. Properties of $\mathcal{C}_n$.

Associated closely with $\{B_n\}$ is the Morgan–Voyce type sequence $\{C_n\}$ defined [8; $x = 1$] by

$$C_n = 3C_{n-1} - C_{n-2} \tag{3.1}$$

with initial conditions

$$C_0 = 2, C_1 = 3. \tag{3.2}$$

Considering the unit coefficient sums $\mathcal{C}_n(\mathcal{C}_0 = 0)$ corresponding to the representation situation for $\mathcal{B}_n$, we have

$$\mathcal{C}_n = \sum_{i=0}^{n-1} C_i \qquad (C_n = L_{2n}[8]), \tag{3.3}$$

$$= L_{2n-1} + 1. \tag{3.4}$$

In fact,

(3.5)

$\mathcal{C}_0$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\cdots$
0	2	5	12	30	77	200	522	$\cdots$

Consequences of this structure flow as hereunder.

(3.6) *Recurrence:* $$\mathcal{C}_n = 3\mathcal{C}_{n-1} - \mathcal{C}_{n-2} - 1.$$

(3.7) *Generating function:* $$\sum_{i=1}^{\infty} \mathcal{C}_i x^{i-1} = (2-3x)[1-(4x-4x^2+x^3)]^{-1}.$$

(3.8) *Binet form:* $$\mathcal{C}_n = \alpha^{2n-1} + \beta^{2n-1} + 1.$$

(3.9) *Simson formula:* $$\mathcal{C}_{n+1}\mathcal{C}_{n-1} - \mathcal{C}_n^2 = \mathcal{C}_n - 6.$$

(3.10) *Summations:* $$\sum_{i=1}^{n} \mathcal{C}_{2i} = F_{4n+1} + n - 1 = \mathcal{B}_{2n} + n,$$

$$\sum_{i=1}^{n} \mathcal{C}_{2i-1} = F_{4n-1} + n - 1 = \mathcal{B}_{2n-1} + n, \tag{3.11}$$

$$\sum_{i=1}^{n} \mathcal{C}_i = L_{2n-1} + n - 2. \tag{3.12}$$

$$(3.13)\ \textit{Other simple properties:}\quad \mathcal{C}_n - \mathcal{C}_{n-1} = L_{2n-2},$$

$$(3.14)\quad \mathcal{C}_n + \mathcal{C}_{n-1} = 5F_{2n-2} + 2,$$

$$(3.15)\quad \mathcal{C}_n - \mathcal{C}_{n-2} = 5F_{2n-3},$$

$$(3.16)\quad \mathcal{C}_n = 2\mathcal{B}_n - 3\mathcal{B}_{n-1},$$

$$(3.17)\quad \mathcal{B}_n + \mathcal{C}_n = 3F_{2n},$$

$$(3.18)\quad \mathcal{B}_n - \mathcal{C}_n = F_{2n-3} - 2,$$

$$(3.19)\quad \mathcal{C}_{-n} = -\mathcal{C}_{n+1} + 2.$$

Following the notation in the *divertissement*, we define

$$(3.1)'\quad \mathcal{C}'_n = \mathcal{C}_{n+1} + \mathcal{C}_{n-1} = 3\mathcal{C}_n - 1 \qquad \text{by (3.6)},$$

giving

$$(3.20)\quad \mathcal{C}'_n = 3\mathcal{C}'_{n-1} - \mathcal{C}'_{n-2} - 2 \qquad (\textit{recurrence})$$

Then

$$(3.21)\quad \mathcal{C}'_n - \mathcal{C}'_{n-1} = 3L_{2n-2},$$

$$(3.22)\quad \mathcal{C}'_n + \mathcal{C}_{n-1} = 15F_{2n-2} + 4,$$

$$(3.23)\quad \mathcal{C}'_n - \mathcal{C}'_{n-2} = 3(\mathcal{C}_n - \mathcal{C}_{n-2}) = 15F_{2n-3},$$

$$(3.24)\quad \sum_{i=1}^{n} \mathcal{C}'_i = 3\sum_{i=1}^{n} \mathcal{C}_n - n,$$

$$(3.25)\quad \mathcal{C}'_{-n} = -\mathcal{C}'_{n+1} + 2.$$

Extending the superscript numeration as for $\mathcal{B}_n$, we arrive at

$$(3.26)\quad \mathcal{C}_n^{(m+1)} = \mathcal{C}_{n+1}^{(m)} + \mathcal{C}_{n-1}^{(m)} \qquad (\mathcal{C}_n^{(0)} = \mathcal{C}_n).$$

A *representation-derived grid* for $\mathcal{C}_n^{(m)}$ may now be constructed. From this, it ensues that

$$(3.27)\quad \mathcal{C}_{-n}^{(m)} = \mathcal{C}_{n+1}^{(m)} + 2^{m+1},$$

$$(3.28)\quad \mathcal{C}_0^{(m)} = 2^m - 3^m = -\mathcal{B}_0^{(m)},$$

$$(3.29)\quad \mathcal{C}_n^{(m)} - \mathcal{C}_{n-1}^{(m)} = 3^m L_{2n-2},$$

$$(3.30)\quad \mathcal{C}_n^{(m)} + \mathcal{C}_{n-1}^{(m)} = 3^m \cdot 5F_{2n-2} + 2^{m+1},$$

$$(3.31)\quad \mathcal{C}_n^{(m)} - \mathcal{C}_{n-2}^{(m)} = 3^m(5F_{2n-3}),$$

$$(3.32)\quad \mathcal{C}_n^{(m)} = 3^m\mathcal{C}_n + \mathcal{C}_0^{(m)} = 3^m L_{2n-1} + 2^m,$$

$$(3.33)\quad \sum_{i=1}^{n} \mathcal{C}_i^{(m)} = 3^m \sum_{i=1}^{n} \mathcal{C}_i + n\mathcal{C}_0^{(m)}.$$

Furthermore, the grid system allows us to infer that,

$$\text{(3.34)} \qquad \text{for rows } (n): \qquad \mathcal{C}_n^{(m)} = 3\mathcal{C}_n^{(m-1)} - 2^{m-1},$$

$$\text{(3.35)} \qquad \text{for columns } (m): \qquad \mathcal{C}_n^{(m)} = 3\mathcal{C}_{n-1}^{(m)} - \mathcal{C}_{n-2}^{(m)} - 2^m.$$

Finally, there are the hybrid results

$$\text{(3.36)} \qquad \mathcal{B}_n^{(m)} + \mathcal{C}_n^{(m)} = 3^m(\mathcal{B}_n + \mathcal{C}_n) = 3^{m+1}F_{2n},$$

$$\text{(3.37)} \qquad \mathcal{B}_n^{(m)} - \mathcal{C}_n^{(m)} = 3^m F_{2n-3} - 2^{2m+1}.$$

4. MIDDLE: $\mathcal{B}_n^*$.

Define the *augmented sequence* $\mathcal{B}_n^*(a,b,k) \equiv \mathcal{B}_n^*$ by

$$\text{(4.1)} \qquad \mathcal{B}_{n+2}^*(a,b,k) = 3\mathcal{B}_{n+1}^*(a,b,k) - \mathcal{B}_n^*(a,b,k) + k$$

with initial conditions

$$\text{(4.2)} \qquad \mathcal{B}_1^*(a,b,k) = a, \quad \mathcal{B}_2^*(a,b,k) = b.$$

Hence

$$\text{(4.3)} \qquad \mathcal{B}_{n+1}^*(0,1,1) = \mathcal{B}_n.$$

The first few members of $\{\mathcal{B}_n^*\}$ are:

(4.4)

$\mathcal{B}_1^*$	$\mathcal{B}_2^*$	$\mathcal{B}_3^*$	$\mathcal{B}_4^*$	$\mathcal{B}_5^*$	$\mathcal{B}_6^*$	$\cdots$
a	b	$3b-a+k$	$8b-3a+4k$	$21b-8a+12k$	$55b-21a+33k$	$\cdots$

Immediately then

$$\text{(4.5)} \qquad \mathcal{B}_n^* = bF_{2n-2} - aF_{2n-4} + k(F_{2n-3} - 1).$$

Calculations readily disclose (4.1), (4.2) that

$$\text{(4.6)} \qquad \mathcal{B}_0^* = 3a - b + k,$$

$$\text{(4.7)} \qquad \mathcal{B}_{-1}^* = 8a - 3b + 4k.$$

Various specializations arise, of which a sample is herewith recorded.

$$\text{(4.8)} \qquad \mathcal{B}_n^*(1,1,1) = 2F_{2n-3} - 1,$$

$$\text{(4.9)} \qquad \mathcal{B}_n^*(1,1,0) = F_{2n-3},$$

$$\text{(4.10)} \qquad \mathcal{B}_n^*(-1,1,0) = L_{2n-3},$$

$$(4.11) \qquad \mathcal{B}_n^*(L_{2n-2}, L_{2n-4}, 2) = 2F_{2n-3} \qquad \text{by } [1;\ I_{28}],$$

$$(4.12) \qquad \sum_{i=1}^{n} \mathcal{B}_n^* = bF_{2n-1} - aF_{2n-3} + k(F_{2n-2} - n + 1) \qquad \text{by } [10],$$

so

$$(4.13) \qquad \sum_{i=1}^{n} \mathcal{B}_i^*(1,1,1) = 2F_{2n-2} - n + 1,$$

and

$$(4.14) \qquad \sum_{i=1}^{n} \mathcal{B}_i^*(-1,1,1) = F_{2n-1} - n + 1.$$

Observe that

$$\begin{aligned}
&(4.15) & \mathcal{B}_n^*(2,5,-1) &= \mathcal{C}_n, \\
&(4.16) & \mathcal{B}_n^*(4,13,2) &= \mathcal{B}'_n \\
&(4.17) & \mathcal{B}_n^*(5,14,-2) &= \mathcal{C}'_n, \\
&(4.18) & \mathcal{B}_n^*(1,2,0) &= b_n \qquad [8,\ x=1], \\
&(4.19) & \mathcal{B}_n^*(1,4,0) &= c_n \qquad [8,\ x=1],
\end{aligned}$$

where $\{b_n\}$ is a second Morgan–Voyce number sequence [8], and $\{c_n\}$ [8] is associated with it. Values of a, b, k for B_n and C_n are readily obtainable.

Other aspects which may be taken up as in [8] are, e.g., determinants, the generating function, the Binet form, and the Simson formula.

Suppose we pursue briefly the concept introduced in the *divertissement*, namely, to define $(\mathcal{B}_n^*)' = \mathcal{B}_{n+1}^* + \mathcal{B}_{n-1}^*, (\mathcal{B}_n^*)'' = (\mathcal{B}_{n+1}^*)' + (\mathcal{B}_{n-1}^*)', \cdots,$

$$(4.20) \qquad (\mathcal{B}_n^*)^{(m+1)} = \left(\mathcal{B}_{n+1}^*\right)^{(m)} + \left(\mathcal{B}_{n-1}^*\right)^{(m)}, \quad (\mathcal{B}_n^*)^{(0)} = \mathcal{B}_n^*.$$

Without much ado, we then derive

$$\begin{aligned}
&(4.21) & (\mathcal{B}_n^*)^{(m)} &= 3\left(\mathcal{B}_{n-1}^*\right)^{(m)} - \left(\mathcal{B}_{n-2}^*\right)^{(m)} + 2^m k, \\
&(4.22) & \mathcal{B}_{-n}^*(a,b,k) &= \mathcal{B}_{n+3}^*(b,a,k),
\end{aligned}$$

so

$$(4.23) \qquad \mathcal{B}_{-n}^*(1,1,k) = \mathcal{B}_{n+3}^*(1,1,k).$$

Properties analogous to those evolved for $\mathcal{B}'_n$, and others, may now be investigated as the spirit moves us.

5. END

Representation number sequences $\mathbf{b}_n, \mathbf{c}_n$ can be constructed from b_n, c_n (4.18), (4.19) in a manner similar to that (1.3), (3.3) for $\mathcal{B}_n, \mathcal{C}_n$ from B_n, C_n, thus creating a quartet of numerical relations. An analysis of the properties of $\mathbf{b}_n, \mathbf{c}_n$ resembling those for $\mathcal{B}_n, \mathcal{C}_n$ is, however, more appropriately the subject of a further offering.

Other possibilities for future development include, e.g., extension to negative subscripts $\mathcal{B}_{-n}, \mathcal{C}_{-n}, \mathbf{b}_{-n}, \mathbf{c}_{-n} (n > 0)$.

Evidence that the minimal representation applicable to $\mathcal{B}_n$ (1.3), (1.5) is unique may be envisaged from the emerging patterns in the abbreviated table provided for integers $N = 1, 2, 3, \cdots, 35$. (Zero coefficients are indicated by an empty space.) Uniqueness is possibly the subject of another project.

N	B_1	B_2	B_3	B_4
	1	3	8	21
1	1			
2	2			
3		1		
4	1	1		
5	2	1		
6		2		
7	1	2		
8			1	
9	1		1	
10	2		1	
11		1	1	
12	1	1	1	
13	2	1	1	
14		2	1	
15	1	2	1	
16			2	
17	1		2	
18	2		2	
19		1	2	
20	1	1	2	
21				1
22	1			1
23	2			1
24		1		1
25	1	1		1
26	2	1		1
27		2		1
28	1	2		1
29			1	1
30	1		1	1
31	2			1
32		1	1	1
33	1	1	1	1
34	2	1	1	1
35		2	1	1

ABBREVIATED MINIMAL REPRESENTATION TABLE FOR
$\{B_n\} : n = 1, 2, 3, 4, 5.$

Notice that the crucial criterion for the minimal representation in $\sum_{i=1}^{\infty} \beta_i B_i \quad (\beta_i = 0, 1, 2)$ is:

Juxtaposition of 2,2 *as successive coefficients in the representation does not occur,*
i.e., $2B_n + 2B_{n+1} \quad (n \geq 2)$ is necessarily excluded.
This is so because $2B_n + 2B_{n+1} - B_{n+2} = B_{n-1}$ (by (1.1), $x = 1$), whence

$$2B_n + 2B_{n+1} \begin{cases} > B_{n+2} & \text{if } n \geq 2 \\ = B_{n+2} & \text{if } n = 1 \ (B_0 = 0, (1.2)), \end{cases}$$

whereas $2B_n + 2B_{n+1} < B_{n+2}$ is required.

Lastly, we comment that an analysis of a table for a maximum representation of positive integers N by means of the B_n reveals a mixture of some specifically maximum representations and some which coincide with the minimal representations. Reading from the Table we see that, e.g.,

N	Min. rep.	Max. rep.	Same/Different
7	1 2	1 2	Same
8	0 0 1	2 2	Different

This feature of the representations underscores the assertion in Section 1 that the $\mathcal{B}_n$ in (1.3) correspond to the MinMax numbers for other sequences, e.g., Pell and Jacobsthal, (but are not the totality of MinMax numbers in the current situation for Morgan-Voyce numbers). Further investigations of these aspects are left to the private entertainment of the reader rather than to a formal enquiry here.

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