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TWO VARIANTS OF THE CONCEPT "LOGARITHMIC PROGRESSION"

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We shall define two variants of the concept "logarithmic progression" and shall show some of their properties and applications.

Let everywhere a, b be given real numbers for which $a \geq b \geq e$ (e is the Napier's number) and k be given natural number.

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The sequence defined by the following scheme

$$M(a, b, 0) = 1$$
$$M(a, b, k+1) = \log_b a . M(a, b, k) \equiv \log_b (a . M(a, b, k))$$

will be called "a multiplicative logarithmic progression".

Here we shall show some properties of this progression.

LEMMA 1: For every $x \geq 1$, for every $\alpha \geq \beta = \dfrac{x.\ln b}{x.\ln b - 1}$:

$$x^\alpha - x \geq \alpha . \log_b x .$$

Proof: Obviously, for every fixed real number $x \geq 1$ follows that $\beta > 1$ and from the inequality

$$e^y > 1 + y$$

for every real number $y > 0$ (see e.g. [1]) it follows that

$$e^z > e.z$$

for every real number $z > 1$. Then

$$x^p - 1 \geq \ln x^p$$

for every real number $p > 0$ and therefore

$$x^\alpha - x = x.(x^{\alpha-1} - 1) \geq x.\ln x^{\alpha-1} = (\alpha - 1).\ln b . \log_b x$$

$$= (\alpha - 1) . \frac{\dfrac{x.\ln b}{x.\ln b - 1}}{\dfrac{x.\ln b}{x.\ln b - 1} - 1} . \log_b x = (\alpha - 1) . \frac{\beta}{\beta - 1} . \log_b x$$

(from the obvious inequality $\dfrac{p}{p - 1} \geq \dfrac{q}{q - 1}$ for every two real numbers $q \geq p > 1$)

$$\geq (\alpha - 1) . \frac{\alpha}{\alpha - 1} . \log_b x = \alpha . \log_b x .$$

<u>THEOREM 1</u>: For every natural number k:

$$M(a, b, k) \leq (\log_b a)^{\frac{\ln a}{\ln a - 1}} .$$

Proof: When $k = 1$, the assertion is valid. Let us assume that it be valid for some $k \geq 1$. Then

$$M(a, b, k+1) = \log_b a . M(a, b, k) = \log_b a + \log_b M(a, b, k)$$

$$\leq \log_b a + \frac{\ln a}{\ln a - 1} . \log_b \log_b a = x + \frac{\ln a}{\ln a - 1} . \log_b x$$

where $x = \log_b a = \dfrac{\ln a}{\ln b} \geq 1$ (i.e. $\ln a = x.\ln b$) and from Lemma 2 it follows that:

$$= x + \frac{x.\ln b}{x.\ln b - 1} . \log_b x \leq x^{\frac{x.\ln b}{x.\ln b - 1}} = x^{\frac{\ln a}{\ln a - 1}} = (\log_b a)^{\frac{\ln a}{\ln a - 1}} .$$

With which the theorem is proved.

Therefore for every two real numbers a, b (for which $a \geq b \geq e$, as it is initially supposed), the sequence M(a, b, k) is increasing and limited. Hence, it has a least upper bound and let

$$LM(a, b) = \lim_{k \to \infty} M(a, b, k).$$

THEOREM 2: For every two real numbers a, b:

$$LM(a, b) = \frac{1}{\ln b}.LM(\frac{a}{\ln b}, e). \qquad (1)$$

Proof: Initially, we shall show that for every $k \geq 1$:

$$\frac{1}{\ln b}.M(\frac{a}{\ln b}, e, k) = M(a, b, k) \qquad (2)$$

When k = 1:

$$\frac{1}{\ln b}.M(\frac{a}{\ln b}, e, 1) = \frac{1}{\ln b}.\ln\frac{a}{\ln b} = \log_b a = M(a, b, 1).$$

Let us assume that (2) be valid for some $k \geq 1$. Then:

$$\frac{1}{\ln b}.M(\frac{a}{\ln b}, e, k+1) = \frac{1}{\ln b}.\ln\frac{a}{\ln b}.M(\frac{a}{\ln b}, e, k)$$

$$= \log_b \frac{a}{\ln b}.M(\frac{a}{\ln b}, e, k)$$

(by assumption)

$$= \log_b a.M(a, b, k) = M(a, b, k+1).$$

The validity of (1) follows from:

$$\frac{1}{\ln b}.LM(\frac{a}{\ln b}, e) = \frac{1}{\ln b}.\lim_{k \to \infty} M(\frac{a}{\ln b}, e, k)$$

$$= \lim_{k \to \infty} \frac{1}{\ln b}.\ln M(\frac{a}{\ln b}, e, k)$$

(from (2))

$$= \lim_{k \to \infty} M(a, b, k) = LM(a, b).$$

* * *

The sequences defined by the following schemes

$A(a, b, 0) = 0$

$A(a, b, k+1) = \log_b a + A(a, b, k) \equiv \log_b(a + A(a, b, k))$

will be called" an "additive logarithmic progression".

By analogy with Theorem 1, the following asserion is proved.

THEOREM 3: For every natural number k:

$$A(a, b, k) \leq \frac{a.\ln b}{a.\ln b - 1}.\log_b a.$$

Proof: When k = 0, the assertion is obvious. Let $k \geq 0$ be a fixed natural number. Then by assumption

$$A(a, b, k+1) = \log_b a + A(a, b, k) \leq \log_b a + \frac{a.\ln b}{a.\ln b - 1}.\log_b a)$$

(from Lemma 1)

$$\leq \frac{a.\ln b}{a.\ln b - 1}.\log_b a.$$

Therefore for every two real numbers a, b the sequence A(a, b, k) is increasing and limited. Hence, it has a least upper bound and let

$$LA(a, b) = \lim_{k \to \infty} A(a, b, k).$$

THEOREM 4: For every two real numbers a, b:

$$LA(a, b) = \frac{1}{\ln b}.LA(a, e). \qquad (3)$$

Proof: Initially, we shall show that for every $K \geq 1$:

$$\frac{1}{\ln b}.A(a,\ e,\ K) = A(a,\ b,\ K) \qquad (4)$$

When $K = 1$ the assertion is checked directly. Let $K \geq 1$. Then

$$\frac{1}{\ln b}.A(a,\ e,\ K+1) = \frac{1}{\ln b}.\ln\ (a + \frac{1}{\ln b}.A(a,\ e,\ K))$$

(by assumption)

$= \log_b a + A(a,\ e,\ K) = A(a,\ e,\ K+1)$.

The validity of (3) follows from:

$$\frac{1}{\ln b}.LA(a,\ e) = \frac{1}{\ln b}.\lim_{K\to\infty} A(a,\ e,\ K) = \lim_{K\to\infty} \frac{1}{\ln b}.A(a,\ b,\ K)$$

(from (4))

$= \lim_{K\to\infty} A(a,\ b,\ K) = LA(a,\ b)$.

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Finally, we shall note two applications of both progressions. For this aim we shall represent the solution of the equation

$$e^x = e + x \qquad (5)$$

in the above forms. It can directly be seen, that it is

$$\mu = -e + e^{-e+e^{-e+e^{\cdots}}} = \ln(e + \ln(e + \ln(e + \ldots)$$
$$= 1.420370118020083\ldots$$

(calculated by a computer) and for it the following assertion is valid.

LEMMA 3: μ is a transcendental number.

Proof: Let us assume, that μ be an algebraical number. But (1) can be writen in the form

$$1.e^{\mu} - 1.e^{1} - \mu.e^{0} = 0,$$

where all coeffitients and exponent sings are algebraical numbers and from Lindemann's theorem (see e.g. [2, 3]) we reach a contradiction.

From above we see that $\mu = LA(e,\ e)$.

On the other hand

$LM(e,\ e) = \ln e.\ln e.\ \ln e\ \ldots = 1 + \ln \ln e.\ln e \ldots$

$= 1 + \ln 1 + \ln \ln 1.\ln e \ldots = 1 + LA(e,\ e)$,

i.e.

$\mu = LM(e,\ e) - 1$.

By analogy, the solution of the equation

$$b^x = a.x$$

is $x = LM(a,\ b)$ and the solution of the equation

$$p^x = q.x + r$$

is $x = LM(p^{r/q}.q,\ p) - \frac{r}{q}$.

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