NNTDM 2 (1996) 4, 52-55

# NEW INTEGER FUNCTIONS, RELATED TO $\varphi$ AND $\sigma$ FUNCTIONS. V

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Here we shall continue the research from [1] and [2]. Initially, we shall note that the first three part of this series are collected in [1]. Paper [2] is their fourth part (in the title of [2] there is a misprint - there the symbol "$\psi$" must be read "$\varphi$").

The basis of our research below are the three well-known arithmetic functions $\varphi$, $\sigma$ and $\psi$, which have the following forms (see, e.g. [3,4]).

For every natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ (where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers, $p_1, p_2, \ldots, p_k$ are different prime numbers):

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1),$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),$$

$$\sigma(n) = \prod_{i=1}^{k} (p_i^{\alpha_i + 1} - 1)/(p_i - 1).$$

Let us apply the substitution from [5] over the above (formal) record of $n$, $\varphi(n)$, $\psi(n)$ and $\sigma(n)$. As it is shown in [5],

$$\frac{+ \;\; . \;\; - \;\; :}{. \;\; + \;\; : \;\; -}(\varphi(n)) = \sum_{i=1}^{k} (\alpha_i - 1).p_i + (p_i : 1) = \sum_{i=1}^{k} \alpha_i.p_i = \varsigma(n),$$

and

$$\frac{+ \;\; . \;\; - \;\; :}{. \;\; + \;\; : \;\; -}(\sigma(n)) = \sum_{i=1}^{k} (((\alpha_i + 1).p_i):1) - (p_i : 1) = \varsigma(n),$$

where function $\varsigma(n) = \sum_{i=1}^{k} \alpha_i.p_i$ is defined in [2,5].

Moreover, we can check similarly, that

$$\frac{+ \;\; . \;\; - \;\; :}{. \;\; + \;\; : \;\; -}(n) = \sum_{i=1}^{k} \alpha_i.p_i = \varsigma(n),$$

and

$$\frac{+ \;\; . \;\; - \;\; :}{. \;\; + \;\; : \;\; -}(\psi(n)) = \sum_{i=1}^{k} (\alpha_i - 1).p_i + (p_i . 1) = \sum_{i=1}^{k} \alpha_i.p_i = \varsigma(n),$$

i.e. after applying of the substitution $\frac{+ \;\; . \;\; - \;\; :}{. \;\; + \;\; : \;\; -}$ over $n$, $\varphi(n)$, $\psi(n)$

and $\sigma(n)$ we obtain the one and the same results - an expression which corresponds to the function $\zeta(n)$.

We must accentuate that this substitution is only a formal procedure and it is not mathematically correct.

Obviously, Dedekind's function $\psi$ is a modification of Euler's function $\varphi$. The question for the modifications of $\sigma$ function is interesting too.

Here we shall define three functions, every one of which is a modification of $\sigma$ function. They are the following

$$\sigma_{+,+}(n) = \prod_{i=1}^{k} (p_i^{\alpha_i+1} + 1)/(p_i + 1),$$

$$\sigma_{+,-}(n) = \prod_{i=1}^{k} (p_i^{\alpha_i+1} + 1)/(p_i - 1),$$

$$\sigma_{-,+}(n) = \prod_{i=1}^{k} (p_i^{\alpha_i+1} - 1)/(p_i + 1),$$

Obviously, we can put

$$\sigma_{-,-}(n) = \sigma(n).$$

We must note, that the above modifications are made over the formal record of $\sigma$ function. The new functions do not have any of the good properties of $\sigma$ functions. For example, they are not integer ones. Their domain is the set of the natural numbers, but their range is the set of the rational numbers. We can transform these functions to integer ones, using as a multiplier one of the following two functions:

$$\text{mult}_{+}(n) = \prod_{i=1}^{k} (p_i + 1)$$

and

$$\text{mult}_{-}(n) = \prod_{i=1}^{k} (p_i - 1),$$

which are analogous of the function $\text{mult}(n) = \prod_{i=1}^{k} p_i$ (see [6]). In this case, the functions

$$\bar{\sigma}_{+,+}(n) = \sigma_{+,+}(n).\text{mult}_{+}(n),$$

$$\bar{\sigma}_{-,+}(n) = \sigma_{-,+}(n).\text{mult}_{+}(n)$$

and

$$\overline{\sigma}_{+,-}(n) = \sigma_{+,-}(n).\text{mult}_{-}(n)$$

are already integer functions. Practically, function $\text{mult}_{+}(n)$ coincides with function $\tau(n)$ from [3].

They have also the following forms:

$$\overline{\sigma}_{+,+}(n) = \prod_{i=1}^{k}(p_i^{\alpha_i+1} + 1),$$

$$\overline{\sigma}_{+,-}(n) = \prod_{i=1}^{k}(p_i^{\alpha_i+1} + 1),$$

$$\overline{\sigma}_{-,+}(n) = \prod_{i=1}^{k}(p_i^{\alpha_i+1} - 1),$$

i.e. $\overline{\sigma}_{+,+}$ and $\overline{\sigma}_{+,-}$ coincide.

Below, we shall discuss the basic properties of the above σ-types of functions. From the definitions, it can easily be seen that the following assertions are valid for every natural number n with the above form.

Theorem 1: (a) $\frac{+\ .\ -\ :}{.\ +\ :\ -}(\sigma_{+,+}(n) = \sum_{i=1}^{k}\alpha_i.p_i = \varsigma(n),$

(b) $\frac{+\ .\ -\ :}{.\ +\ :\ -}(\sigma_{-,+}(n) = \sum_{i=1}^{k}\alpha_i.p_i = \varsigma(n),$

(c) $\frac{+\ .\ -\ :}{.\ +\ :\ -}(\sigma_{+,-}(n) = \sum_{i=1}^{k}\alpha_i.p_i = \varsigma(n).$

Theorem 2: $\varphi(n) < \sigma_{-,+}(n) < \sigma_{+,+}(n) < n < \psi(n) \le \sigma(n) < \sigma_{+,-}(n).$

Theorem 3: (a) $\varphi(n).\sigma(n) = \sigma_{-,+}(n).\psi(n),$

(b) $\varphi(n).\sigma_{+,-}(n) = \sigma_{+,+}(n).\psi(n),$

(c) $\sigma_{+,-}(n).\sigma_{-,+}(n) = \sigma_{+,+}(n).\sigma(n),$

(d) $\varphi(n).\sigma_{\pm,-}(n) = \sigma_{\pm,\pm}(n).n.\text{mult}_{\pm}(n),$

(e) $\psi(n).\sigma_{\pm,+}(n) = \sigma_{\pm,\pm}(n).n.\text{mult}_{\pm}(n).$

where the sigh "±" denotes that the index of the new type of σ-function can be as "+", as well as "-".

Theorem 4: $\varphi(n).\psi(n) \le \varphi(n).\sigma(n) < \psi(n).\sigma(n) < n^2 <$

$< \varphi(n).\sigma_{+,-}(n) < \sigma_{-,+}(n).\sigma(n) < n.\psi(n).$

Theorem 5: (a) $\frac{+\ .\ -\ :}{.\ +\ :\ -}(\sigma_{+,+}(n) = \sum_{i=1}^{k}(\alpha_i + 1).p_i = \varsigma(n) + \text{sum}_1(n),$

$$(b)\ \frac{+}{.}\,\frac{.}{+}\,\frac{-}{:}\,\frac{:}{-}\,(\overline{\sigma}_{-,+}(n) = \sum_{i=1}^{k} (\alpha_i + 1).p_i = \varsigma(n) + sum_1(n),$$

$$(c)\ \frac{+}{.}\,\frac{.}{+}\,\frac{-}{:}\,\frac{:}{-}\,(\overline{\sigma}_{+,-}(n) = \sum_{i=1}^{k} (\alpha_i + 1).p_i - \varsigma(n) + sum_1(n),$$

where $sum_1(n) = \sum_{i=1}^{k} p_i$ (see [6]).

Theorem 6: $\varphi(n) < \overline{\sigma}_{-,+}(n) < n < \psi(n) < \sigma(n) < \overline{\sigma}_{+,-}(n)$.

Theorem 7: $\overline{\sigma}_{+,-}(n).\overline{\sigma}_{-,+}(n) = \overline{\sigma}_{+,+}(n).\sigma(n).mult(n)$.

Theorem 8: All new defined functions are multiplicative ones.

Finally, we shall note that all new defined functions can be generalized in the sense of [7-9].

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Received in BNT in Jan. 1994