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A GENERALIZATION OF AN ARITHMETIC FUNCTION

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Following the idea from [1], we can define for two natural numbers $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ (where $k, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers, $p_1, p_2, \ldots, p_k$ are different prime numbers) and $s \geq 1$:

$$\partial_s(n) = \begin{cases} 1, & \text{if } s > k \\ \sum\limits_{(i_1,\ldots,i_s)\in S} \alpha_{i_1}.\ \ldots\ .\alpha_{i_s}.p_1^{\alpha_1}.\ \ldots\ .p_{i_1-1}^{\alpha_{i_1-1}}.p_{i_1}^{\alpha_{i_1}-1}. \\ \quad p_{i_1+1}^{\alpha_{i_1+1}}.\ \ldots\ .p_{i_s-1}^{\alpha_{i_s-1}}.p_{i_s}^{\alpha_{i_s}-1}.p_{i_s+1}^{\alpha_{i_s+1}}.\ \ldots\ .p_{i_k-1}^{\alpha_{i_k}}, \\ \quad \text{if } s \leq k \end{cases}$$

where $S = \{(i_1, i_2, \ldots, i_s) \ /\ 1 \leq i_1 < i_2 < \ldots < i_s \leq k\} \subset N^s$ (N is the set of the natural numbers).

Let n and s have the above forms everywhere.

From the above definition the validity of the following two assertions follow directly.

THEOREM 1:

$$\partial_s(n) = \begin{cases} 1, & \text{if } s > k \\ n.\sum\limits_{(i_1,\ldots,i_s)\in S} \dfrac{\alpha_{i_1}.\alpha_{i_2}.\ \ldots\ .\alpha_{i_s}}{p_{i_1}.p_{i_2}.\ \ldots\ .p_{i_s}}, & \text{if } s \leq k \end{cases}$$

THEOREM 2: For every $t \leq s$ in number prime numbers $p_1, p_2, \ldots, p_t$:

$$\partial_s(p_1.p_2.\ \ldots\ .p_t) = 1.$$

THEOREM 3: For every natural number $m \geq 1$:

$$\partial_s(n^m) = n^{m-1}.m^s.\partial_s(n).$$

Proof: $\partial_s(n^m) = \partial_s((\prod_{i=1}^{k} p_i^{\alpha_i})^m) = \partial_s(\prod_{i=1}^{k} p_i^{m.\alpha_i})$

$$= \begin{cases} 1, & \text{if } s > k \\ n^m.\sum\limits_{(i_1,\ldots,i_s)\in S} \dfrac{(m.\alpha_{i_1}).(m.\alpha_{i_2}).\ \ldots\ .(m.\alpha_{i_s})}{p_{i_1}.p_{i_2}.\ \ldots\ .p_{i_s}}, & \text{if } s \leq k \end{cases}$$

$$= \begin{cases} 1, & \text{if } s > k \\ n^m.m^s.\sum\limits_{(i_1,\ldots,i_s)\in S} \dfrac{\alpha_{i_1}.\alpha_{i_2}.\ \ldots\ .\alpha_{i_s}}{p_{i_1}.p_{i_2}.\ \ldots\ .p_{i_s}}, & \text{if } s \leq k \end{cases}$$

$= n^{m-1}.m^s.\partial_s(n).$

On the other hand, the equality $\partial_s(m.n) = \partial_s(m).n + m.\partial_s(n)$ is not valid for $s > 1$ although. For example, if $m = a.b$ and $n = c.d$, where a, b, c and d are different prime numbers, then

$\partial_s(m.n) = \partial_s(a.b.c.d) = a.b + a.c + a.d + b.c + b.d + c.d$

$> a.b + c.d = \partial_s(m).n + m.$

The function ∂_s can be interpreted as a function for (so to say simultaneous) differentiating.

Obviously, when $s = 1$, we obtain the analogical result from [1].

REFERENCE:

[1] Atanassov K. New integer functions, related to φ and σ functions, Bull. of Number Theory and Related Topics, Vol. XI (1987), No. 1, 3-26.

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