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ON THE SOLUTIONS OF $t_m + t_n = t_x$ AND $t_m - t_n = t_y$

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The problem of determining all non-trivial (i.e. $m, n, x, y \geq 1$) solutions of the system

$$\begin{cases} t_m + t_n = t_x & (1) \\ t_m - t_n = t_y, & (2) \end{cases}$$

where $t_k = k.(k + 1)/2$ is the k-th triangular number, is formulated in Waclaw Sierpinski's monography [1] on the triangular numbers. Following the method from [2] and the results from [3], we shall determine the set of all solutions of (1) and (2).

Initially, we note, that the set of solutions $\langle x, m, n \rangle$ of (1) is determined in [3]. It is

$$\{\langle (b.(b + 1) + r^2 - r)/2.r,\ (b.(b + 1) - r^2 - r)/2.r,\ b\rangle :$$
$$(b \in N)\ \&\ (r \in \square(b.(b + 1))\ \&\ E\text{-}(r,\ b.(b + 1)/r)\ \&\ (\ 2 \leq r \leq b - 1)\},$$

where (see [3]): $\square(n) = \{m : m / n \text{ and } m^2 \leq n\}$.

From (1) and (2) we obtain:

$$t_x - t_y = 2.t_n. \qquad (3)$$

The set of the solutions $\langle x, y, n \rangle$ of (3) is constructed by the same means as it is done in [3] and this set is the following:

$$\{\langle (2.a.(a + 1) + q^2 - q)/2.q,\ (2.a.(a + 1) - q^2 - q)/2.q,\ a\rangle :$$
$$(a \in N)\ \&\ (q \in \square(2.a.(a + 1))\ \&\ E\text{-}(q,\ 2.a.(a + 1)/q)\ \&\ (\ 2 \leq q \leq a.\sqrt{2})\}.$$

For obtaining of the solutions $\langle x, y, m, n \rangle$ of the system (1)-(2) we must replace:

$$a = b$$

and

$$(2.a.(a + 1) + q^2 - q)/2.q = (b.(b + 1) + r^2 - r)/2.r,$$

i.e.

$$q.r^2 - (2.a.(a + 1) + q^2).r + a.(a + 1).q = 0,$$

from where

$$f(a, q) = r = (2.a.(a + 1) + q^2 - \sqrt{4.a^2.(a + 1)^2 + q^4})/2.q \quad (4)$$

The second root does not satisfy the conditions abour q and r, because

$$(2.a.(a + 1) + q^2 + \sqrt{4.a^2.(a + 1)^2 + q^4})/2.q > 2.a.(a + 1)/q$$
$$> a + 1.$$

Let predicate P(a,r) note that r = f(a, q) from (4) is a natural number.

From:

$$(b.(b + 1) - r^2 - r)/2.r = (b.(b + 1) + r^2 - r)/2.r - r$$
$$= (2.a.(a + 1) + q^2 - q)/2.q - f(a, q)$$

follows that all components of the system can be determined.

From (2) follows that m > n, i.e.

$$(b.(b + 1) - r^2 - r)/2.r > b$$

or

$$(2.a.(a + 1) + q^2 - q)/2.q - f(a, q) > a.$$

Hence:

$$f(a, q) < (2.a.(a + 1 - q) + q^2 - q)/2.q.$$

Therefore all solutions <x, y, m, n> of the system (1) - (2) are elements of the set

$$\{<(2.a.(a + 1) + q^2 - q)/2.q, \ (2.a.(a + 1) - q^2 - q)/2.q,$$
$$(2.a.(a + 1) + q^2 - q)/2.q - f(a, q), a> : (a \in N) \ \&$$
$$(q \in \square(2.a.(a + 1)) \ \& \ E-(q, a.(a + 1)/q) \ \& \ (2 \leq q \leq$$
$$a.\sqrt{2}) \ \& \ P(a, q) \ \& \ E-(f(a, q), 2.a.(a + 1)/f(a, q))$$
$$\& \ (f(a, q) < (2.a.(a + 1 - q) + q^2 - q)/2.q)\}.$$

By condition, a ≥ 2.

Let a = 2, then q = 2, but P(a, q) is not valid.

Let a = 3, then q = 3, but P(a, q) is not valid.

Let $a = 4$, then $q = 4$, but $P(a, q)$ is not valid.

Let $a = 4$, then $q = 5$, but $P(a, q)$ is not valid.

Let $a = 5$, then $q = 2$, but $P(a, q)$ is not valid.

Let $a = 5$, then $q = 3$, but $P(a, q)$ is not valid.

Let $a = 5$, then $q = 5$ and $r = 2$. Then $\langle 8, 3, 6, 5\rangle$ is an element of the above set (see [1]) and this is the minimal one in this set about the value of a.

Another question is discussed in [1], too: to determine the solutions of the system (1)-(2), where $m = n + 1$.

From the above form of the set follows directly, that the answer is related to the solution of the equality:

$$(m =)\ (2.a.(a + 1) + q^2 - q)/2.q = f(a, q) = a + 1 \ (= n + 1),$$

i.e.

$$2.a.(a + 1) + q^2 - q - 2.q.f(a, q) = 2.q.(a + 1)$$

and from (4) it follows that

$$(2.a + 3)^2.q^2 = 4.a^2.(a + 1)^2 + q^4. \qquad (5)$$

Let the predicate $Q(a, q)$ be valid iff a and q are solutions of (5).

Therefore, the set of the solutions $\langle x, y, n\rangle$ of the system

$$\begin{cases} t_{n+1} + t_n = t_x \\ t_{n+1} - t_n = t_y \end{cases}$$

is

$$\{\langle (2.a.(a + 1) + q^2 - q)/2.q,\ (2.a.(a + 1) - q^2 - q)/2.q,\ a\rangle : (a \in N)\ \&\ (q \in D(2.a.(a + 1)))\ \&\ E\text{-}(q, a.(a + 1)/q)\ \&\ (2 \le q \le a.\sqrt{2})\ \&\ Q(a, q)\}.$$

REFERENCES:

[1] Sierpinski W., Liczbi trojkatne, Panstwowe Zaklady Wydawnictw Szkolnych, Warszawa (in Polish).

[2] Atanassov K., A set-method for representation of the solutions of some Diophantine equations and some of its applications, see pp. 21-26.

[3] Atanassov K., On the solutions of $t_x = t_y + t_z$. Bulletin of Number Theory, Vol. XVI, 1992, 57-59.

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