NNTDM 2 (1996) 4, 21-26

A SET-METHOD FOR REPRESENTATION OF THE SOLUTIONS OF SOME DIOPHANTINE EQUATIONS AND SOME OF ITS APPLICATIONS

Krassimir T. Atanassov
Math. Research Lab., P.O.Box 12, Sofia-1113, BULGARIA

Different methods in number theory, related to the solution of Diophantine equations, determine as a result a list of all solutions of such a given equation, or information, that a considered equation has no solutions. Below we shall introduce a method which is based on some elements of the set theory. The solutions are given as sets of natural numbers, which satisfy some conditions. These conditions must be described by some recursive and calculable functions or predicates. One elementary example is the following: the set of the solutions of the Diophantine equation

$$x^2 + y^2 = z^2 \qquad (1)$$

is the set of three-tuples, corresponding to x, y and z, respectively:

$$\{\langle (a^2 - q^2)/2.q,\ a,\ (a^2 + q^2)/2.q\rangle,\ \langle a,\ (a^2 - q^2)/2.q,$$
$$(a^2 + q^2)/2.q\rangle\ :\ (a \in N)\ \&\ (q \in \Box(a^2))\ \&\ (E{+}(q,\ a^2/q))\},$$

where N is the set of the natural numbers,

$$\Box(\alpha) = \{\beta:\ (\exists\gamma \in N)(\alpha = \beta.\gamma)\ \&\ (\beta^2 \le \alpha)\}$$

and E+(a,b) is the predicate, which notes that a and b are with the same evenness and predicate E-(a,b) notes that a and b are with different evenness.

The elements of this set are obtained through the following procedure:

1. It is fixed an arbitrary natural number a;
2. It is replaced y = a;
3. It is transformed (1) to the system with a form:

$$(z - x).(z = x) = a^2 \qquad (2)$$

4. It is determine an arbitrary natural number q such that $q \in \Box(a^2)$ and $E{+}(q,\ a^2/q)$;
5. It is transformed (2) to the form:

$$\begin{cases} z - x = q \\ z + x = a^2/q \end{cases}$$

6. It is determined the elements of the first member of the above set, i.e.,

$$\langle (a^2 - q^2)/2.q,\ a,\ (a^2 + q^2)/2.q\rangle.$$

The elements of the second type members of this set are obtained through substitution of x with y in 2, 3 and 5 from the above procedures.

Through similar procedure we can solve other Diophantine equations, also. It is important to mark, that the solutions of (1), obtained by other methods, can be represented by the solutions of (1) obtained through the above method.

Actually, it is well known that the solutions of (1) can be represented in the form:

$$x = u^2 - v^2,\ y = 2.u.v,\ z = u^2 + v^2$$

or

$$x = 2.u.v,\ y = u^2 - v^2,\ z = u^2 + v^2,$$

for some natural numbers u and v for which (u, v) = 1 and E-(u, v) (see e.g., [1]). Let u > v and let a = 2.u.v and $q = 2.v^2$. Then

$$(a^2 - q^2)/2.q = (4.u^2.v^2 - 4.v^4)/4.v^2 = u^2 - v^2.$$

The other cases are checked analogically.

On the other hand, the three-tuple <448, 840, 952> is a solution of (1) and it can be represented by the above method as follows: $a = 840 = 2^3.3.5.7$, $q = 504 = 2^3.3^2.7$, but u and v, which can represent the components of the three-tuple do not exist, because these components have a common divisor $2^3.7$; there exist u and v for which (1) has a solution <8, 15, 17> and the first one is obtained from the second one.

* * *

The author does not know publications in which this method is used. Some of its elements can be seen in the author's papers [2-4].

Now we shall apply the method to some Diophantine equations and thus we shall illustrate it.

In [5] a list of 21 open problems is given. The problem 10 is the following: "Is there a rectangular parallelepiped for which all arcs and diagonals of the lateral faces are natural numbers, i.e. is the Diophantine equation

$$x^2 + y^2 + 2.z^2 = u^2 + v^2 \qquad (3)$$

resolvable in natural numbers?"

Below we shall comment especially the formulation of the problem following [6]. Initially, we shall determine (by the above method) the solutions of (3). Let a, b and c be three arbitrary (fixed) numbers for which the following relation is valid:

$$n = a^2 + 2.b^2 - c^2 > 1$$

and if n is an even number, let 4 be a divisor of n.

Let

$$\begin{cases} x = a \\ z = b \\ u = c. \end{cases} \qquad (4)$$

Therefore (3) contains the form:

$$a^2 + y^2 + 2.b^2 = c^2 + v^2,$$

i.e.

$$(v - y).(v + y) = n.$$

Let $q \in \square(n)$. Then

$$\begin{cases} v - y = q \\ v + y = n/q \end{cases}$$

and hence

$$\begin{cases} v = (n + q^2)/2.q \\ y = (n - q^2)/2.q \end{cases} \qquad (5)$$

Directly it is seen, that (4) and (5) are the solutions of (1), i.e. the solutions of (3) are the elements of the set

$A = \{\langle a,\ (a^2 + 2.b^2 - c^2 - q^2)/2.q,\ b,\ c,\ (a^2 + 2.b^2 - c^2 + q^2)/2.q\rangle : (a,\ b,\ c \in N)\ \&\ (2 \in \mathcal{D}(a^2 + 2.b^2 - c^2)$ iff $4 \in \mathcal{D}(a^2 + 2.b^2 - c^2))\ \&\ (q \in (a^2 + 2.b^2 - c^2))\ \&\ E{+}(q,\ (a^2 + 2.b^2 - c^2)/q)\}$.

The set A is infinite, because a, b and c are three arbitrary numbers. For example, two elements of A are the following: <1, 1, 2, 1, 3>, <3, 3, 4, 5, 5>. Let us discuss the geometrical interpretation of the problem. The solutions <3, 3, 4, 5, 5> can be interpreted as lengths of the arcs of a rectangular parallelepiped ABCDEFGH (AB, CD, EF and GH are parallel; AE, BF, CG and DH - also; AD, BC, EH and FG - also) for which the lengths of AB, BC, BF, AF and CF are respectively the above five values.

On the other hand, the values 1, 1, 2, 1, 3 cannot be the lengths of arcs of any rectangular parallelepiped. Therefore the text in [5] is not correct. The algebraic interpretation of the problem 10 from [5] is related to the system with the following two Diophantine equations:

$$\begin{cases} x^2 + z^2 = u^2 \\ y^2 + z^2 = v^2 \end{cases} \qquad (6)$$

Below we shall construct the set of all solutions of (6).

Let b be a fixed natural number. Let:

$$z = b.$$

Then (8) can be reorganized to:

$$\begin{cases} (u - x).(u + x) = b^2 \\ (v - y).(v + y) = b^2 \end{cases} \qquad (7)$$

Let $q,\ r \in \mathcal{D}(b)$ and let $E{+}(b,q)$, $E{+}(b,r)$. If b is a prime number, then $q = r = 1$. From (7) we obtain that:

$$x = \frac{b^2 - q^2}{2.q},\quad y = \frac{b^2 - r^2}{2.r},\quad u = \frac{b^2 + q^2}{2.q},\quad v = \frac{b^2 + r^2}{2.r},$$

i.e. the solutions are elements of the set

$\{\langle \frac{b^2 - q^2}{2.q},\ \frac{b^2 - r^2}{2.r},\ b,\ \frac{b^2 + q^2}{2.q},\ \frac{b^2 + r^2}{2.r}\rangle : (b \in N)\ \&\ (q,\ r \in \mathcal{D}(b))\ \&\ E{+}(b,q)\ \&\ E{+}(b,r)\}$.

Obviously, these values of x, y, z, u and v satisfy (6).

We will go beyond the theme of the paper, mentioning that other (equivalent) solutions of the above problem are given in [7]. There the following problem is formulated, too. "Is there a rectangular parallelepiped for which all arcs and diagonals of the lateral and basis faces are natural numbers, i.e., is the system of Diophantine equations

$$\begin{cases} x^2 + y^2 = u^2 \\ x^2 + z^2 = v^2 \\ y^2 + z^2 = w^2 \end{cases}$$

solvable?" (see also [8]).

Similar (but non-solved) problem is the following: "Is there a rectangular parallelepiped for which all arcs and diagonals of the lateral faces and the body diagonal are natural numbers, i.e., is the system of Diophantine equations

$$\begin{cases} x^2 + z^2 = u^2 \\ y^2 + z^2 = v^2 \\ x^2 + y^2 + z^2 = t^2 \end{cases}$$

solvable?"

The most general (non-solved) problem is the following: "Is there a rectangular parallelepiped for which all arcs and diagonals of the lateral faces and basis and the body diagonal are natural numbers, i.e., is the system of Diophantine equations

$$\begin{cases} x^2 + y^2 = u^2 \\ x^2 + z^2 = v^2 \\ y^2 + z^2 = w^2 \\ x^2 + y^2 + z^2 = t^2 \end{cases}$$

solvable?"

* * *

Another problem, which can be solved by the above method is the following (see [9]): to determine the solutions of the Diophantine equantion

$$n = x^2 + y^2 - z^2, \tag{8}$$

where

$$x^2 \le n, \ y^2 \le n, \ z^2 \le n,$$

and x, y, z, l, m, n are natural numbers.

Let n and a be fixed natural numbers, where a has the following properties:

1. $a^2 \le n$,
2. a is an odd number, when 4 is not divisor of n, but 2 is divisor of n, or when 4 is divisor of $n - a^2$ for an odd number n; a is an even number, when 4 is divisor of n or when n is an odd number.

Let $m = n - a^2$. Then (8) obtains the form:

$$y^2 - z^2 = m. \tag{9}$$

Let $q \in \Box(m)$ and $E+(m,q)$. Let, it be $m = 2^{k+1}.s$, then $q = 2^{k}.r$ (r, s – odd numbers). Then (9) can be represented as

$$\begin{cases} y - z = q \\ y + z = \frac{m}{q}, \end{cases}$$

from where the solutions n, x, y and z of (8) are elements of the set

$$\{\langle n,\ a,\ \frac{m + q^2}{2.q},\ \frac{m - q^2}{2.q}\rangle : (a,\ n \in N)\ \&\ (m = n - a^2)\ \&\ (a^2 \le n)\ \&$$
$$(E-(a) \equiv ((E+(n)\ \&\ E-(n/2))\ \vee\ (E-(n)\ \&\ D(4,\ n - a^2)))\ \&$$
$$(E+(a) \equiv (E-(n)\ \vee\ (D(4,\ n))))\},$$

where $E+(x)$, $E-(x)$ and $D(x, y)$ are predicates which note that x is an even number, x is an odd number and that x is a divisor of y, respectively.

* * *

A third example for the application of the set-method is related to the Heronus triangular problem (see e.g. [10, 11]). In [12] it is given a particular solution which we shall adduce here. The problem is the following: "To determine all Heronus triangulars whose sides and surface are integer numbers, i.e., to solve in integer numbers the Diophantine equation:

$$u.v.w.(u + v + w) = s^2." \qquad (10)$$

Let a and b be fixed natural numbers and let

$$\begin{cases} u = a^2 \\ v = b^2 \end{cases}$$

Let $q \in \Box(a^2 + b^2)$. Then a set of solutions $\langle u,\ v,\ w,\ s\rangle$ (but it is possible, that this set does not contain all solutions) of (10) has the form:

$$\{\langle a^2,\ b^2,\ (\frac{a^2 + b^2 - q^2}{2.q})^2,\ a.b.\frac{(a^2 + b^2)^2 - q^4}{4.q^2}\rangle : (a,\ b \in N)\ \&$$
$$(q \in \Box(a^2 + b^2))\}.$$

This set cannot be a fully one, because there are restrictions for the forms of u and v (they are squares). The formulas of the lenghts of the sides of the arbitrary Heronus triangulars are given in [13] (the authos knows only the paper's abstract). Using our method, we can describe the set of these sides in the following form:

$$\{\langle((a.c.g)^2 + (d.f.h)^2).b.e.i/k,\ (a.b.c^2 - d.e.f^2).(a.e.g^2 +$$
$$b.d.h^2).i/k,\ ((e.f.g)^2 + (b.c.h)^2).a.d.i/k\rangle : (a,\ b,\ c,\ d,\ e,\ f,$$
$$g,\ h,\ i \in N)\ \&\ (a.b.c^2 - d.e.f^2 > 0)\ \&\ (k = GCD(a.b.c^2 - d.e.f^2,$$
$$a.e.g^2 + b.d.h^2))\}.$$

Such aa described and illustrated method can be applied to different other Diophantine equations, too.

In a next paper we shall apply this method for constructing the set of the solutions of the system of Diophantine equations:

$t_m + t_n = t_x$ and $t_m - t_n = t_y$,

where $t_k = k.(k + 1)/2$ is the k-th triangular number.

REFERENCES:

[1] Nagell T., Introduction to number theory, John Wiley & Sons, Inc., New York.

[2] Atanassov K., One extremal problem. 2., Bull. of Number Theory and Related Topics Vol. IX (1985), No. 2, 11-13.

[3] Atanassov K., Vasilev M. An extremal problem. 3. Bull. of Number Theory and Related Topics Vol. X (1986), No. 1, 19-25.

[4] Atanassov K., On the solutions of $t_x = t_y + t_z$, Bull. of Number Theory and Related Topics, Vol. XVI, 1992, 57-59.

[5] Kudrevatov G., A book of problems in number theory, Nauka (Moscow), 1970 (in Russian).

[6] Atanassov K., On one problem for a rectangular parallelepiped with integer sides and its modifications, Proc. of Sci. Session of VIVU, W. Tarnovo, 1988, 146-150 (in Bulgarian).

[7] Lal M., Blundon W., Solutions of Diophantine equantions $x^2 + y^2 = l^2$, $y^2 + z^2 = m^2$, $z^2 + x^2 = n^2$, Math. Comput., 1966, Vol. 20, No. 93, 144-147.

[8] Colman W., Some observations on the classical cuboid and its parametric solutions, The Fib. Quar., 1988, Vol. 26, No. 3, 338-343.

[9] Chiao K., On Diophantine equation $n = x^2 + y^2 - z^2$, $x^2 \leq n$, $y^2 \leq n$, $z^2 \leq n$, Acta scient. natur., Univ. Szechuan, 1959, No. 6, 1-10 (in Chinese; I know only the review).

[10] Bradis V., Sicikov A., Despa triunghiurile heronice, Gaz. mat. si. fiz. 1969, A11, No. 6, 325-334.

[11] Battaglia A., Formule parametriche per triangoli heroniami, Archimede, 1959, Vol. 11, No. 3, 163-167.

[12] Atanassov K., On Heronus triangular problem and its modifications, Proc. of Sci. Session of VIVU, W. Tarnovo, 1988, 151-155.

[13] Popovich K., Heronus triangulars, Rev. math. pures et appl., Vol. 7, 1962, No. 3, 439-457.

Received in BNT in Oct. 1992