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ON SEQUENCES OF COMPOUND BRAIDS — SOME PROPERTIES AND PROBLEMS

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Nature uses only the longest threads to weave her patterns, so each small piece of her fabric reveals the organisation of the entire tapestry. Richard Feynman
(in Genius, p.13, J. Glieck, Abacus)

1. INTRODUCTION

This paper introduces a type of closed braid which is a compound of two cylindrical braids bridged by a flat braid (we call these CFC-Braids). We shall treat only the special case where the two cylindrical braids are identical; these we call CFC*-Braids. Such a braid can be described also (and equivalently) as a cylindrical braid with a hole occurring centrally within it. When viewed in this way, we call them CWH*-Braids.

First we shall give a general description of CWH*-Braids, using grid-diagrams (see [1],[2],[3],[5]) to define them. Then we shall investigate four particular classes of CWH*-Braid string-runs. Each class will be defined by specifying the special kind of behaviour which its members display around the hole in the braid.

Equations and parameter-conditions which determine the dimensions of braids in these classes will be presented. The method of deriving these equations and conditions will not be described fully; but brief indications will be given, sufficient for the reader to extract them from the diagrams that accompany them.

Then we shall show how the basic Fibonacci integer sequence may be used in various ways to generate infinite sequences of solutions for members in three of the classes. We shall tabulate a satisfyingly complete set of all such solutions which provide the four values for (H,V,X,B), which mark the dimensions of the Hole and the Cylinder, and which four values belong to some general Fibonacci recurrence. Again we shall not give proofs of the entries in the solutions Table; though not difficult, it would take many pages to present them in full detail.

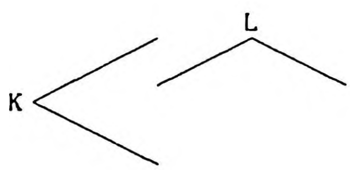
2. CYLINDRICAL BRAIDS WITH HOLES

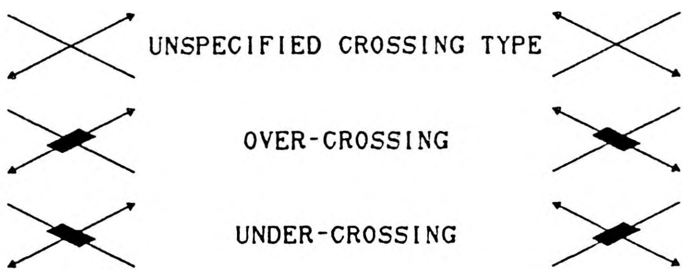
First we must introduce some concepts and terms.

A cylindrical braid string-run [1] is formed by passing one or more strings over the surface of a cylinder, in two diagonal directions, between the left and right circular boundaries of the cylinder. Each string in due course returns to its starting point. Such a closed string path will be called a string-polygon. If one string-polygon does not complete the string-run of the cylindrical braid, then another is started and finished; and so on, until the whole string-run is completed. We shall use the symbol # to denote the number of string-polygons in a given string-run.

Descriptive terms are needed for two kinds of 'point' which occur in a string-run, namely **bight points** and **crossing points**. Whenever a moving string changes direction, it forms a **bight point**, just at the turn; when it crosses another string, it forms a **crossing point**, where the two strings cross.

The following diagrams show bight points (labelled K,L on the left); and our conventions for indicating string crossings (on the right). Since we shall not be discussing weaving patterns in this paper, we shall only use the unspecified type of crossing point.





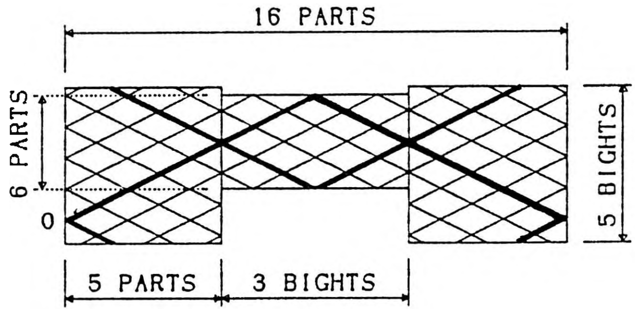
Bight Points at K and L *Crossing Points*

A cylindrical braid with B bight-points (on each circular boundary) has a cylindrical length which we 'measure' by the number of Parts P, or unit steps counted in a pass between the circular boundaries (see [2]). The total number of crossing points in the grid-diagram of a cylindrical braid (without a hole) is easily shown to be $B(P-1)$.

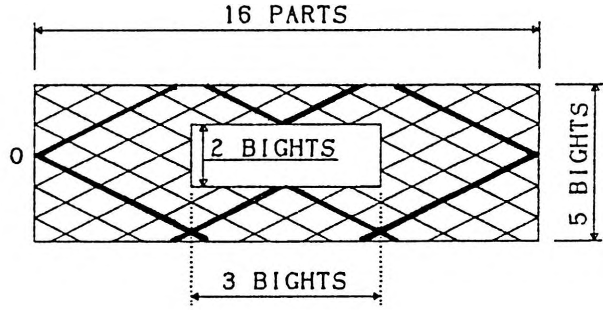
Grid-diagrams for a CWH*-Braid

If we take two identical cylindrical braids, and bridge them by a flat braid, we obtain a special CFC-Braid, namely a CFC*(or CWH*)-Braid. These are the only kinds of CFC*-Braid to be studied in this paper. The left-hand diagram below shows an example, in CFC* form. A string-polygon is shown upon it, in heavy lines.

The diagram on the right shows the equivalent CWH*-Braid.



Grid-Diagram for a CFC-Braid (showing one string-polygon only)*



Equivalent CWH-Braid (parameters (H, V,X,B))*

Parameters of a CWH*-BRAID

As the right-hand diagram shows, four parameters are needed to specify a CWH*-Braid, namely (H, V, X, B), where $2X$ is the number of parts of the whole cylindrical length, B is the number of bights on each circular boundary; and H and V are the numbers of bights on the horizontal and vertical Hole boundaries respectively.

Thus for the above example, $(H, V, X, B) = (3, 2, 8, 5)$.

Note that a CWH*-Braid must have a cylindrical length of an even number (2X) of Parts to enable the Hole to be centrally placed.

Some Problems for Study

The fundamental problem, for all types of braid, is to determine whether or not they can be braided with one continuous string. In general, one wishes to determine the number(#) of string-polygons that are needed to complete the string-run. This number is a fundamental property of the braid.

Then, for any given string-run one can study the dispositions of the string-polygon or string-polygons, as they are laid down on the cylinder-with-hole. For example, one can study their 'shapes'; we shall see below that for the CWH*-Braid classes of this paper, each member has a certain number of string-polygons, and that the sets of these string-polygons partition into two or three equivalence classes.

As a historical note, string-polygons for flat braids, without holes, have been studied as paths of billiard balls on a table with reflecting boundaries (see [4]), and are known as König-Szücs polygons. Kronecker(1823-1891) wrote on their global arithmetic behaviour, placing the work in the field of Diophantine Approximations.

A more general problem is to allow the four parameters H, V, X, B to vary according to some fixed rules, maintaining constant relationships between the parameters. Any set of rules will serve to define a class of the Braids; within this class one can, for example, look for single-string members, and study their properties.

One can also, for example, look at all the 2-string members of a class, and determine how the pairs of closed strings interact. It is very instructive to use computer software to carry out exploratory studies. When viewing the braiding process on a computer-screen, the moving strings can be observed as they fill the grid-diagram, and the string polygons can be drawn with different colors; then the intersecting string-polygons form interesting colored weave-patterns. Geometrical patterns can be discerned; and arrangements of bight formations around the Hole can be observed, which are distinctive to particular string-polygons. Attempts may be made to classify these geometric weave-patterns, and the associated Hole bight formation arrangements, in mathematical ways that are amenable to analysis.

3. FOUR SIMPLE CLASSES OF CWH*-BRAIDS

We shall now define four classes of CWH*-Braids, and study their mathematical properties. They are particularly simple ones, since their members have string-polygons which have easily defined and understood bight patterns around the Hole. Nevertheless, these braids have many interesting algebraic and pattern properties; and they serve as a good introduction to the theory of CWH*-Braids.

Definitions

(i) Two string-polygons are **equivalent** if they can be traced simultaneously in such a way that edges met along the way correspond in parallel pairs. (N.B. This is an equivalence relation, which partitions the set of string-polygons of a given braid into equivalence classes.)

(ii) Three types of string-polygon are:

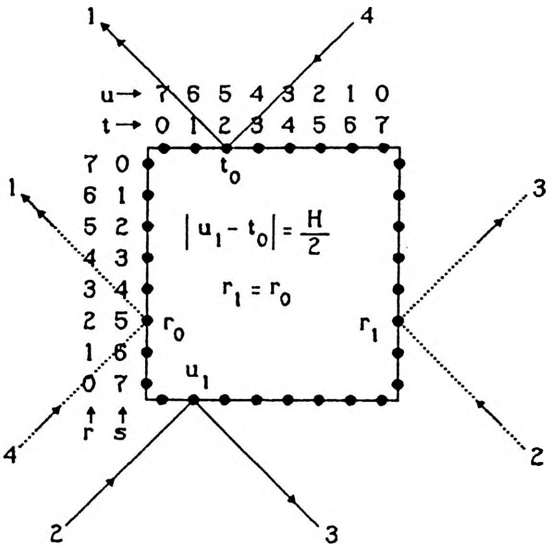
Type-A: The string-polygon fills just one bight-point on each horizontal Hole boundary.

Type-B: The string-polygon fills just one bight-point on each vertical Hole boundary. (It can be proved that these two bight-points are horizontally opposed across the Hole.)

Type-C: The string-polygon fills just one bight-point on one vertical Hole boundary only.

The four classes

It can be shown that the above types of string-polygons serve to define just four classes of CWH*-Braids. These are as follows.



Class I: contains only string-polygons of Type-A and Type-B, where for a Type-A string-polygon the following condition applies (see the diagram): $|u_1 - t_0| = H/2$.

$H = (n_1+1)X - m_1B$ $\qquad n_1 = \mathrm{ODD}$

$V = m_2B - (n_2+1)X + (u_1 - t_0)$ $\qquad n_2 = \mathrm{EVEN}$

$V = m_3B - (n_3+1)X - (u_1 - t_0)$ $\qquad n_3 = \mathrm{EVEN}$

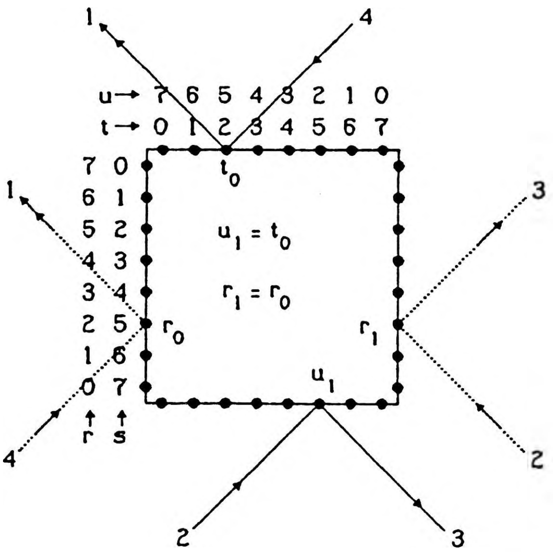
$2B = 2(n_1+1)V + (n_2+n_3+2)H$

$2X = 2m_1V + (m_2+m_3)H$ $\qquad n_1 + 1 = n_2 - n_3$

$(n_2+1)m_3 - (n_3+1)m_2 = 1$ $\qquad m_1 = m_2 - m_3$

H=EVEN

V=EVEN OR ODD



Class II contains only string-polygons of Type-A and Type-B, where for a Type-A string-polygon the following condition applies (see the diagram): $u_1 = t_0$.

$H = (n_1+1)X - m_1B$ $\qquad n_1 = \mathrm{ODD}$

$V = m_2B - (n_2+1)X$ $\qquad n_2 = \mathrm{EVEN}$

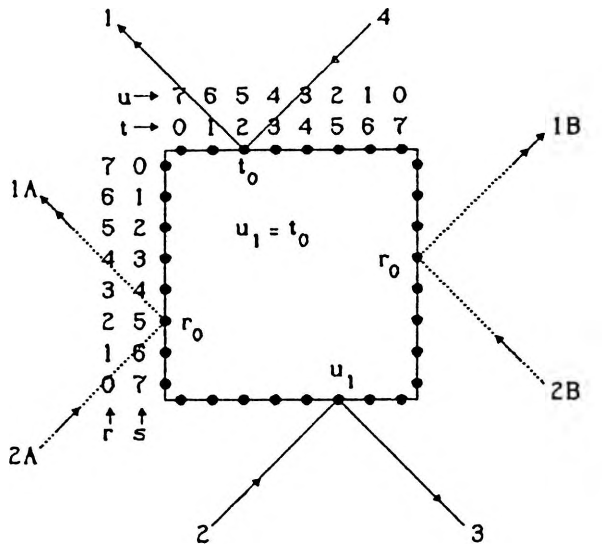
$B = (n_1+1)V + (n_2+1)H$

$X = m_1V + m_2H$

$(n_1+1)m_2 - (n_2+1)m_1 = 1$

H=EVEN OR ODD $\qquad m_1 = \mathrm{ODD}$

V=EVEN OR ODD



Class III contains only string-polygons of Type-A and Type-C, where for a Type-A string-polygon the following condition applies (see the diagram): $u_1 = t_0$.

$H=(n_1+1)X-m_1B$ $\quad n_1=\text{EVEN}$

$V=m_2B-(n_2+1)X$ $\quad n_2=\text{EVEN}$

$B=(n_1+1)V+(n_2+1)H$

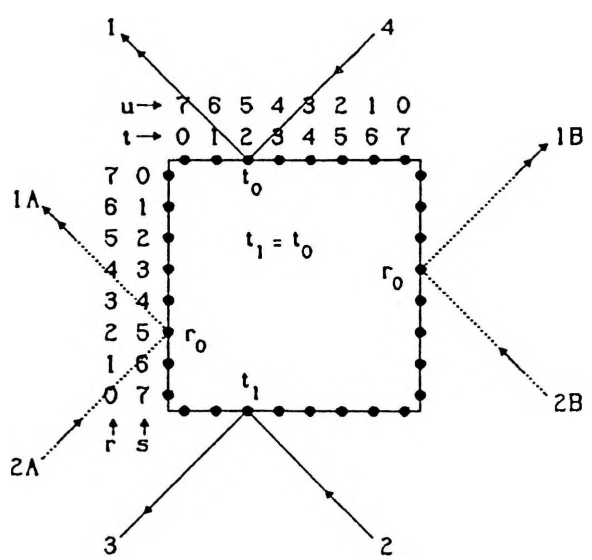
$X=m_1V+m_2H$

$(n_1+1)m_2-(n_2+1)m_1=1$

H=EVEN OR ODD

V=EVEN OR ODD

m_1 AND m_2 OF DIFFERENT PARITY



Class IV contains only string-polygons of Type-A and Type-C, where for a Type-A string-polygon the following condition applies (see the diagram): $t_1 = t_0$.

$H=(n_1+1)X-m_1B$ $\quad n_1=\text{EVEN}$

$V=m_2B-(n_2+1)X$ $\quad n_2=\text{ODD}$

$B=(n_1+1)V+(n_2+1)H$

$X=m_1V+m_2H$

$(n_1+1)m_2-(n_2+1)m_1=1$

H=EVEN OR ODD

V=EVEN OR ODD

$m_2=\text{ODD}$

Observation

It is clear from the above class-definitions that the total number (#) of string-polygons in a braid string-run is equal to $H+V$ for a braid in Class I or Class II; and is equal to $H+2V$ for a braid in Class III or Class IV.

Equations relating (H, V, X, B)

The equations presented above, under each of the class-diagrams, show that for each of the four types of Hole 'geometries' to occur, the pairs (H,V) and (X,B) (which determine respectively the Hole and the Cylinder dimensions) are linearly related.

In these equations the integer variables m and n indicate the following:

> m counts the number of times that the string-run of a single string-polygon, running between consecutive boundaries, makes a complete revolution around the cylinder. We call this a winding number of that string-polygon. Note that a total string-polygon may have two winding numbers.
>
> n, similarly, counts the number of times that the string-run of a single string-polygon, running between consecutive boundaries, oscillates between the left-hand and right-hand circular boundaries of the whole cylinder.

The H and V equations are derived by considering the string-runs of the string-polygons which run between the vertical and horizontal Hole boundaries respectively.

The particular Case to be studied

We shall now treat a Case where the CWH*-Braids are such that their parameters (H, V, X, B) are all members of either the general Fibonacci recurrence sequence F(H,V) or of F(V,H), wherein the initiating elements H and V can be any pair of positive integers. The Fibonacci sequence requirements demand that the variables m and $(n+1)$ shall all be Fibonacci integers in the sequence 1,1,2,3,... Also, the B and X equations tell us that the coefficients (n_1+1) and (n_2+1) must be adjacent Fibonacci integers; similarly, so must be the coefficients m_1 and m_2.

It can be shown that only the CWH*-Braid Classes II, III and IV contain such braids, fulfilling all these conditions.

A common constraint in all these classes is the following:

$$\begin{vmatrix} m_2 & m_1 \\ n_2+1 & n_1+1 \end{vmatrix} = 1$$

And since the elements of this determinant must be Fibonacci integers, we know that they must satisfy either of the following two identities:

$$F_nF_{n+2} - F_{n+1}{}^2 = +1 \qquad \text{or} \qquad F_nF_{n+3} - F_{n+1}F_{n+2} = +1.$$

Furthermore, in any particular class, all relevant conditions on these elements must be satisfied. For a braid to be in Class II, for example, parity considerations require that the elements in the determinant have parities thus: $\begin{pmatrix} odd & odd \\ odd & even \end{pmatrix}$ or $\begin{pmatrix} even & odd \\ odd & even \end{pmatrix}$

4. TABULATED SOLUTION SET

The following table gives full details of all the possible solutions for the Case studied.

$F_nF_{n+2}-F_{n+1}{}^2 = +1$		$F_nF_{n+3}-F_{n+1}F_{n+2} = +1$
CLASS II : $o \equiv odd, e \equiv even$; $i = 0, 1, 2, \cdots$		
$\begin{pmatrix} o & o \\ o & e \end{pmatrix}$	$\begin{pmatrix} e & o \\ o & e \end{pmatrix}$	$\begin{pmatrix} e & o \\ o & e \end{pmatrix}$
(*i*) $\begin{pmatrix} F_{1+6i} & F_{2+6i} \\ F_{2+6i} & F_{3+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,3,4) in sequence F(H,V)	(*ii*) $\begin{pmatrix} F_{3+6i} & F_{2+6i} \\ F_{4+6i} & F_{3+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,4,5) in sequence F(V,H)	(*i*) $\begin{pmatrix} F_{3+6i} & F_{4+6i} \\ F_{5+6i} & F_{6+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,5,7) in sequence F(H,V)
(*iii*) $\begin{pmatrix} F_{5+6i} & F_{4+6i} \\ F_{4+6i} & F_{3+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,6,5) in sequence F(V,H)	(*iv*) $\begin{pmatrix} F_{3+6i} & F_{4+6i} \\ F_{2+6i} & F_{3+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,5,4) in sequence F(H,V)	(*ii*) $\begin{pmatrix} F_{6+6i} & F_{5+6i} \\ F_{4+6i} & F_{3+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,7,5) in sequence F(V,H)
CLASS III		
$\begin{pmatrix} e & o \\ o & o \end{pmatrix}$	$\begin{pmatrix} o & e \\ o & o \end{pmatrix}$	$\begin{pmatrix} e & o \\ o & o \end{pmatrix}$
(*i*) $\begin{pmatrix} F_{3+6i} & F_{4+6i} \\ F_{4+6i} & F_{5+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,5,6) in sequence F(H,V)	(*ii*) $\begin{pmatrix} F_{7+6i} & F_{6+6i} \\ F_{8+6i} & F_{7+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,8,9) in sequence F(V,H)	(*i*) $\begin{pmatrix} F_{3+6i} & F_{2+6i} \\ F_{5+6i} & F_{4+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,4,6) in sequence F(V,H)

$(iii)\ \begin{pmatrix} F_{3+6i} & F_{2+6i} \\ F_{2+6i} & F_{1+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,4,3) in sequence F(V,H) $(iv)\ \begin{pmatrix} F_{5+6i} & F_{6+6i} \\ F_{4+6i} & F_{4+5i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,7,6) in sequence F(H,V)	$(ii)\ \begin{pmatrix} F_{4+6i} & F_{5+6i} \\ F_{2+6i} & F_{3+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,6,4) in sequence F(H,V)
CLASS IV $\begin{pmatrix} o & e \\ e & o \end{pmatrix}$ $\begin{pmatrix} o & o \\ e & o \end{pmatrix}$ $(i)\ \begin{pmatrix} F_{5+6i} & F_{6+6i} \\ F_{6+6i} & F_{7+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,7,8) in sequence F(H,V) $(ii)\ \begin{pmatrix} F_{5+6i} & F_{4+6i} \\ F_{6+6i} & F_{5+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,6,7) in sequence F(V,H) $(iii)\ \begin{pmatrix} F_{7+6i} & F_{6+6i} \\ F_{6+6i} & F_{5+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,8,7) in sequence F(V,H) $(iv)\ \begin{pmatrix} F_{7+6i} & F_{8+6i} \\ F_{6+6i} & F_{7+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,9,8) in sequence F(H,V)	$\begin{pmatrix} o & o \\ e & o \end{pmatrix}$ $(i)\ \begin{pmatrix} F_{1+6i} & F_{2+6i} \\ F_{3+6i} & F_{4+6i} \end{pmatrix}$ (H,V,X,B) at positions (1,2,3,5) in sequence F(H,V) $(ii)\ \begin{pmatrix} F_{4+6i} & F_{3+6i} \\ F_{2+6i} & F_{1+6i} \end{pmatrix}$ (H,V,X,B) at positions (2,1,5,3) in sequence F(V,H)

To give but one example, let us take the first solution set in the table, namely that for Case II(i). Setting $i = 0$ we find the equations (which satisfy all the conditions stated for the Case II Hole geometry) relating the dimensions of Cylinder and Hole thus:

$$\begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix} \text{ gives } \begin{pmatrix} X \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} H \\ V \end{pmatrix}$$

Then $X = H + V$ and $B = H + 2V$; and so the Hole and Cylinder dimensions are the first four terms in the general Fibonacci recurrence $F(H, V) = H,\ V,\ (H + V),\ (H + 2V), \cdots$

5. SUMMARY

In this paper we have treated a special case of Cylinder-With-Hole Braids.

The particular Hole-geometry involved resulted in a linear relationship between (H, V) and (X, B). There are, of course, many other relationships, linear and non-linear, which arise for CWH*-Braids when other Hole-geometries are considered.

Within the classes resulting from the particular chosen Hole-geometry, we gave conditions and equations, and tabulated all solutions, for the cases where H, V, X and B must lie within a general Fibonacci recurrence.

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