NNTDM 2 (1996), 2, 40-46

THE UNITARY ANALOGUE OF PILLAI'S ARITHMETICAL FUNCTION II.

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Abstract. Let k be a positive integer and let $(a,b)_{*,k}$ denote the greatest k-th power divisor of a which is a unitary divisor of b. We introduce the function

$$P_k^*(n) = \sum_{i=1}^{n^k} (i, n^k)_{*,k}$$

and obtain the arithmetical evaluation of it and an asymptotic formula for the summatory function of P_k^*, which improves for $k = 1$ an earlier result of the author.

AMS Subject Classification: 11A25, 11N37

Key Words and Phrases: unitary divisor, Pillai's function, arithmetical expression, asymptotic formula

1. Introduction

The arithmetical function

$$P(n) = \sum_{i=1}^{n} (i, n),$$

where (i,n) denotes the g.c.d. of i and n, was introduced by S. S. Pillai [6]. It is easy to show that

$$P(n) = \sum_{de=n} d\phi(e), \tag{1.1}$$

where ϕ is Euler's totient function and formula (1.1) furnishes the asymptotic estimate (see [8])

$$\sum_{n \le x} P(n) = \frac{3}{\pi^2} x^2 \log x + O(x^2). \tag{1.2}$$

Let k be a positive integer and let $(a,b)_k$ denote the largest common k-th power divisor of a and b. H. G. Kopetzky [4] investigated the following generalization of Pillai's function

$$P_k(n) = \sum_{i=1}^{n^k} (i, n^k)_k = \sum_{1 \le i_1, \ldots, i_k \le n} (i_1, \ldots, i_k, n)^k,$$

where $(i_1, \dots, i_k, n)$ stands for the g.c.d. of $i_1, \dots, i_k$ and n. He deduced an asymptotic formula for the summatory function of P_k, which was improved by J. CHIDAMBARASWAMY and R. SITARAMACHANDRARAO [1] showing that

$$(1.3) \qquad \sum_{n \leq x} P_k(n) = \frac{x^{k+1}}{(k+1)\zeta(k+1)} \left(\log x + 2C - \frac{1}{k+1} - \frac{\zeta'(k+1)}{\zeta(k+1)} \right) + O(x^{k+a+\varepsilon}),$$

for every $\varepsilon > 0$, where ζ denotes the Riemann zeta function, C is the Euler constant and a is the exponent appearing in Dirichlet's divisor problem $(1/4 \leq a < 1/3)$, i.e. for every $\varepsilon > 0$

$$\sum_{n \leq x} \tau(n) = x \log x + (2C - 1)x + O(x^{a+\varepsilon}),$$

where $\tau(n)$ is the number of divisors of n.

In the proof of formula (1.3) the authors of paper [1] used the identity

$$P_k(n) = \sum_{de=n} \mu(d) e^k \tau(e),$$

where μ is the Möbius function. For $k = 1$ formula (1.3) gives a considerably sharper estimate for the summatory function of Pillai's arithmetical function than (1.2).

The author of the present paper introduced in [9] the unitary analogue of Pillai's function by

$$P^*(n) = \sum_{i=1}^{n} (i, n)_*,$$

where $(i, n)_*$ is the greatest divisor of i which is a unitary divisor of n. We recall that d is a unitary divisor of n (notation $d || n$) if $d | n$ and $(d, n/d) = 1$. Using that

$$P^*(n) = \sum_{d || n} d\phi^*(n/d),$$

where ϕ^* is the unitary analogue of Euler's function, see E. COHEN [2], we deduced the asymptotic estimate

$$(1.4) \qquad \sum_{n \leq x} P^*(n) = \frac{3\alpha}{\pi^2} x^2 \log x + O(x^2),$$

where

$$(1.5) \qquad \alpha = \prod_p \left(1 - \frac{1}{(p+1)^2} \right),$$

the product being extended over all primes p.

For a fixed positive integer k let $(a,b)_{*,k}$ denote the greatest k-th power divisor of a which is a unitary divisor of b. In this paper we define the function

$$P_k^*(n) = \sum_{i=1}^{n^k} (i, n^k)_{*,k},$$

representing the unitary analogue of the function P_k and obtain the arithmetical evaluation of it and an asymptotic formula for the summatory function of P_k^*, which is analogous to (1.3) and is sharper for $k = 1$ than the estimate (1.4). Our method is elementary and is based on the convolutional identity (2.3) and on formula (3.1).

2. Arithmetical properties

Let μ^* denote the unitary analogue of the Möbius function given by $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n and let $\phi_k^*(n)$ denote the number of integers i in a complete residue system mod n^k such that $(i, n^k)_{*,k} = 1$. It is known, see K. Nageswara Rao [5], that

$$\phi_k^*(n) = \sum_{d||n} d^k \mu^*(n/d). \tag{2.1}$$

Proposition 1.

$$P_k^*(n) = \sum_{d||n} d^k \phi_k^*(n/d). \tag{2.2}$$

Proof. Group the numbers $i \in \{1, \ldots, n^k\}$ according to the value $(i, n^k)_{*,k} = d^k$, where $d^k | i, d||n$ and $(i/d^k, n^k/d^k)_{*,k} = 1$. Hence $i = jd^k, (j, (n/d)^k)_{*,k} = 1, 1 \le j \le (n/d)^k$ and we obtain

$$P_k^*(n) = \sum_{d||n} d^k \sum_{\substack{1 \le j \le (n/d)^k \\ (j,(n/d)^k)_{*,k}=1}} 1 = \sum_{d||n} d^k \phi_k^*(n/d).$$

Proposition 2.

$$P_k^*(n) = \sum_{d||n} d^k \tau^*(d) \mu^*(n/d), \tag{2.3}$$

where $\tau^*(n)$ denotes the number of the unitary divisors of n.

Proof. Let $\times$ denote the unitary convolution, see E. Cohen [2], and let $E_k(n) = n^k$, for every $n \ge 1$. Then using (2.2) and (2.1) we get

$$P_k^* = E_k \times \phi_k^* = E_k \times E_k \times \mu^* = E_k \tau^* \times \mu^*.$$

Corollary 1. *The function P_k^* is multiplicative and if $n = p_1^{a_1}...p_r^{a_r}$, then $P_k^*(n) = (2p_1^{ka_1}-1)...(2p_r^{ka_r}-1)$.*

Proof. Using identity (2.2) the function P_k^* is the unitary convolution of two multiplicative functions, hence it is multiplicative.

Proposition 3. *Another representation of the function P_k^* is given by*

$$P_k^*(n) = \sum_{1\le i_1,\dots,i_k\le n} ((i_1,\dots,i_k),n)_*^k, \tag{2.4}$$

Proof. Group the vectors $(i_1,\dots,i_k)$ with $1 \le i_1,\dots,i_k \le n$ according to the value $((i_1,\dots,i_k),n)_* = d$, where $d|(i_1,\dots,i_k), d||n$ and $((i_1,\dots,i_k)/d, n/d)_* = 1$. Hence $i_1 = j_1 d,\dots, i_k = j_k d, ((j_1,\dots,j_k),n/d)_* = 1, 1 \le j_1,\dots,j_k \le n/d$ and we get

$$\sum_{1\le i_1,\dots,i_k\le n} ((i_1,\dots,i_k),n)_*^k = \sum_{d||n} d^k \sum_{\substack{1\le j_1,\dots,j_k\le n/d\\ ((j_1,\dots,j_k),n/d)_*=1}} 1 = \sum_{d||n} d^k J_k^*(n/d),$$

where J_k^* is the unitary analogue of the Jordan totient function and $J_k^* = \phi_k^*$, see [5], which completes the proof.

3. Asymptotic formulae

We need the following results.

Proposition 4.. (M. V. Subbarao and D. Suryanarayana [7]) *If u is a positive integer, then*

$$\sum_{\substack{n\le x\\ (n,u)=1}} \tau^*(n) = \frac{\phi(u)x}{\zeta(2)\psi(u)}\left(\log x + 2C - 1 + 2\alpha(u) - 2\beta(u) - 2\frac{\zeta'(2)}{\zeta(2)}\right) \tag{3.1}$$

$$+O(S(u)\sqrt{x}\log x),$$

where the O-estimate is uniform in x and u,

$$\psi(u) = u\prod_{p|u}(1+\frac{1}{p})$$

stands for the Dedekind function and

$$\alpha(u) = \sum_{p|u}\frac{\log p}{p-1}, \quad \beta(u) = \sum_{p|u}\frac{\log p}{p^2-1}, \quad S(u) = \sum_{d|u}\frac{3^{\omega(d)}}{\sqrt{d}}. \tag{3.2}$$

Remark. According to [3, p.154] and [7, p.5] we have

$$\alpha(u) = O(\log\log 3u) = O(\log u), \quad \beta(u) = O(1). \tag{3.3}$$

By partial summation we immediately have

Proposition 5. *If k and u are positive integers, then*

$$(3.4)\quad \sum_{\substack{n\le x\\(n,u)=1}} n^k\tau^*(n)=\frac{\phi(u)x^{k+1}}{(k+1)\zeta(2)\psi(u)}\left(\log x+2C-\frac{1}{k+1}+2\alpha(u)-2\beta(u)-2\frac{\zeta'(2)}{\zeta(2)}\right)$$

$$+O(S(u)x^k\sqrt{x}\log x),$$

where the O-estimate is uniform in x and u.

Proposition 6. *If k is a positive integer, then the series*

$$\sum_{n=1}^{\infty}\frac{\mu^*(n)\phi(n)}{n^{k+1}\psi(n)}$$

is absolutely convergent and its sum is given by

$$(3.5)\qquad \alpha_k=\prod_p\left(1-\frac{p-1}{(p+1)(p^{k+1}-1)}\right),$$

the product being extended over all primes p.

Proof. The absolute convergence follows at once by

$$\left|\frac{\mu^*(n)\phi(n)}{n^{k+1}\psi(n)}\right|\le\frac{1}{n^{k+1}}\le\frac{1}{n^2}.$$

The general term is a multiplicative function in n and expanding the series into an infinite product of Euler type we obtain (3.5).

Proposition 7. *If k is a positive integer, then*

$$(3.6)\qquad \sum_{n\le x}\frac{S(n)}{n^k\sqrt{n}}=O(1),$$

where $S(n)$ is defined by (3.2).

Proof.

$$\sum_{n\le x}\frac{S(n)}{n^k\sqrt{n}}=\sum_{de=n\le x}\frac{3^{\omega(e)}}{d^k\sqrt{d}e^{k+1}}=\sum_{d\le x}\frac{1}{d^k\sqrt{d}}\sum_{e\le x/d}\frac{3^{\omega(e)}}{e^{k+1}}$$

$$=\sum_{d\le x}\frac{1}{d^k\sqrt{d}}.O(1)=O\left(\sum_{d\le x}\frac{1}{d^k\sqrt{d}}\right)=O(1),$$

since $k\ge 1$ and $3^{\omega(n)}=O(n^{\varepsilon}),\varepsilon>0$.

Our main result is the following

Theorem. *If k is a positive integer, then*

$$(3.7)\quad \sum_{n\le x} P_k^*(n) = \frac{x^{k+1}}{(k+1)\zeta(2)}\left(\alpha_k\left(\log x + 2C - \frac{1}{k+1} - 2\frac{\zeta'(2)}{\zeta(2)}\right) - A_k + 2B_k - 2C_k\right)$$

$$+O(x^k\sqrt{x}\log x),$$

where α_k *is given by* (3.5) *and*

$$A_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)\log n}{n^{k+1}\psi(n)},$$

$$B_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)\alpha(n)}{n^{k+1}\psi(n)},\quad C_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)\beta(n)}{n^{k+1}\psi(n)}.$$

Proof. By (2.3), (3.4) and (3.3) we get

$$\sum_{n\le x} P_k^*(n) = \sum_{\substack{de=n\le x\\(d,e)=1}} \mu^*(d)e^k\tau^*(e) = \sum_{d\le x}\mu^*(d)\sum_{\substack{e\le x/d\\(e,d)=1}} e^k\tau^*(e)$$

$$= \sum_{d\le x}\mu^*(d)\left(\frac{\phi(d)x^{k+1}}{(k+1)\zeta(2)\psi(d)d^{k+1}}\left(\log\frac{x}{d} + 2C - \frac{1}{k+1} + 2\alpha(d) - 2\beta(d) - 2\frac{\zeta'(2)}{\zeta(2)}\right)\right.$$

$$\left. +O\left(S(d)\left(\frac{x}{d}\right)^{k+\frac{1}{2}}\log\frac{x}{d}\right)\right)$$

$$= \frac{x^{k+1}}{(k+1)\zeta(2)}\left(\left(\sum_{d\le x}\frac{\mu^*(d)\phi(d)}{d^{k+1}\psi(d)}\right)\left(\log x + 2C - \frac{1}{k+1} - 2\frac{\zeta'(2)}{\zeta(2)}\right) - \sum_{d\le x}\frac{\mu^*(d)\phi(d)\log d}{d^{k+1}\psi(d)}\right.$$

$$\left. +2\sum_{d\le x}\frac{\mu^*(d)\phi(d)\alpha(d)}{d^{k+1}\psi(d)} - 2\sum_{d\le x}\frac{\mu^*(d)\phi(d)\beta(d)}{d^{k+1}\psi(d)}\right) + O\left(x^k\sqrt{x}\log x\sum_{d\le x}\frac{S(d)}{d^{k+\frac{1}{2}}}\right)$$

$$= \frac{x^{k+1}}{(k+1)\zeta(2)}\left(\alpha_k\left(\log x + 2C - \frac{1}{k+1} - 2\frac{\zeta'(2)}{\zeta(2)}\right) + O\left(\log x\sum_{d>x}\frac{1}{d^{k+1}}\right)\right.$$

$$\left. -A_k + O\left(\sum_{d>x}\frac{\log d}{d^{k+1}}\right) + 2B_k + O\left(\sum_{d>x}\frac{\log d}{d^{k+1}}\right) - 2C_k + O\left(\sum_{d>x}\frac{1}{d^{k+1}}\right)\right)$$

$$+O\left(x^k\sqrt{x}\log x\sum_{d\le x}\frac{S(d)}{d^{k+\frac{1}{2}}}\right).$$

Now using (3.6) and the well known estimates

$$\sum_{d>x}\frac{1}{d^{k+1}} = O\left(\frac{1}{x^k}\right),\quad \sum_{d>x}\frac{\log d}{d^{k+1}} = O\left(\frac{\log x}{x^k}\right)$$

the proof is complete.

Corollary 2. $(k = 1)$ *We have*

$$\sum_{n \leq x} P^*(n) = \frac{3}{\pi^2} x^2 \left(\alpha \left(\log x + 2C - \frac{1}{2} - 2\frac{\zeta'(2)}{\zeta(2)} \right) - A_1 + 2B_1 - 2C_1 \right) + O(x\sqrt{x} \log x), \tag{3.8}$$

where α *is given by* (1.5).

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