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A NEW POINT OF VIEW ON DIRICHLET'S THEOREM FOR THE PRIME NUMBERS DISTRIBUTION IN AN ARITHMETIC PROGRESSION

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Let the sequence $a_1, a_2, a_3, \ldots$ be an infinite arithmetic progression, where $a_1, a_2, \ldots \in N - \{0\}$, where N is the set of the natural numbers. Then this sequence contains infinitely many primes iff a_1 and $d = a_2 - a_1$ have no common divisor greater than 1.

The above statement is known as a famous Direchlet's theorem for the prime numbers distribution in an arithmetic progression (see [1]).

Let me rewrite the sequence $a_1, a_2, a_3, \ldots$ of the form

$$a,\ a + d,\ a + 2d,\ \ldots \tag{1}$$

where $a = a_1$ and $d = a_2 - a_1$. It is obvious that the terms of (1) coincides with the values of the linear function

$$f(x) = a + dx, \tag{2}$$

when the argument x runs the set of the natural numbers.

As I have observed, the Dirichlet's theorem admits the following unexpected equivalent form:

<u>THEOREM 1</u>: Let $a, d \in N - \{0\}$ be natural numbers. The linear function $f(x)$, given by (2), takes infinitely many primes as its values, when the argument x runs the set of natural numbers, iff the same function takes only two different primes as its values for two different natural values of x.

The proof of the equivalence of both forms of the theorem is evident and I omit it.

The linear function is a polynom of power 1. So I put the question, when can a polynom of degree n, with integer coefficients take infinitely many primes as its values if the argument of the polynom runs the set of natural numbers. Theorem 1 gives me a key to the following hypothesic answer, formulated as

<u>THEOREM 2</u>: Let f(x) be a polynom of degree n with integer coefficients. Then f(x) takes infinitely many primes as its values, when x runs the set of natural numbers, iff there exist n + 1 different natural numbers x_1, x_2, ..., x_{n+1} such that all numbers: $f(x_1)$, $f(x_2)$, ..., $f(x_{n+1})$ are different primes.

As a corollary from the above theorem, in the case of n = 1, I obtain Theorem 1 again, i.e., Dirichlet's theorem, too. But I must note that Theorem 2, as an independent statement, is so far still unproved, so it is only a valuable hypothesis.

Let me remind you the basic algebra theorem, discovered by D'Alembert and perfectly proved by Gauss. It states that every polynom has at least one zero over the complex number field. As a corollary of this theorem there follows:

<u>THEOREM 3</u>: Let f(x) be arbitrary polynom of degree n. Then f(x) has infinitely many zeros (i.e. f(x) is identically zero) iff it has n + 1 different zeros.

It is not difficult to see that between Theorem 2 and Theorem 3 there is a remarkable analogy. Namely the behaviour of primes in Theorem 2 is just the same as that of the polynomial zeroes in Theorem 3. Probably it is not just a happenstance and there is some deep reason to that!

REFERENCES:

[1] <u>Dirichlet P. G. L., Dedekind R.</u>, Vorlesungen uber Zahlentheorie, Chelsea, New York, 1968.