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IS THE RIEMANN HYPOTHESIS PROVED SINCE 100 YEARS AGO?

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With the words of the title I mean to say: it seems that with the formulas known 100 years ago, the truth of the Riemann hypothesis (that all the imaginary zeros of the zeta function have real part 1/2), can be derived in an easy and immediate way.

Let us permit to the reader to form his own concept.

We proceed to develop the subject chronologically.

<u>Year 1859</u>

Riemann publishes his famous memoir. Between other formulas, he derives the following one:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2} + \frac{s(s-1)}{1}\int_1^\infty (x^{-s/2-1/2} + x^{s/2-1})\psi(x)dx$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$.

Putting there $s = 1/2 + it$, we obtain:

$$\xi(\frac{1}{2}+it) = \frac{1}{2} - (t^2 + \frac{1}{4})\int_1^\infty \psi(x)x^{-3/4}\cos(\frac{t}{2}\log x)dx$$

from which follows the surprising conclusion that $\xi(\frac{1}{2} + it)$ is a real quantity for real t.

This property is mentioned in every textbook on the subject; but what every mathenatician (including I myself) has failed to see, up to the present, is that this property involves a very definite assertion about the zeros on the critical line.

Let us perform together the count, that is very simple.

First of all, from its definition:

$$\xi(\frac{1}{2}+it) = -(\frac{t^2 + 1}{8})\pi^{-1/4-it/2}\Gamma(\frac{1}{4} + i\frac{t}{2})\zeta(\frac{1}{2} + it).$$

Now, if a product of complex numbers equals a real number, then we must have:

$$\text{argument of product} = \pm 2k_1\pi$$

if the real number is positive; or

$$\text{argument of product} = \pi \pm 2k_2\pi$$

if the real number is negative.

So we can write

$$\text{argument of product} = \pm k_3\pi$$

in every case.

When this is applied to the formula for $\xi(\frac{1}{2}+it)$ written above, we obtain that

$$\arg\{-(\frac{t^2+1}{8})\pi^{-1/4-it/2}\Gamma(\frac{1}{4}+i\frac{t}{2})\zeta(\frac{1}{2}+it)\} = \pm k_4\pi.$$

This implices:

$$\arg\{-(\frac{t^2+1}{8})\pi^{-1/4}\} + \arg\{\pi^{-it/2}\} + \arg\{\Gamma(\frac{1}{4}+i\frac{t}{2})\} + \arg\{\zeta(\frac{1}{2}+it)\} = \pm k_4\pi.$$

Here we have:

$$\arg\{\frac{}{8}(\frac{t^2+1}{})\pi^{-1/4}\} = \pi_5 \pm 2k\ \pi,$$

$$\arg\{\pi^{-it/2}\} = \arg\{e^{-i.t/2.\log\ \pi}\} = \frac{-t}{2}.\log\ \pi \pm 2k_6\pi.$$

Consequently follows:

$$\arg\{\pi^{-it/2}\} + \arg\{\Gamma(\frac{1}{4}+i\frac{t}{2})\} + \arg\{\zeta(\frac{1}{2}+it)\} = \pm k_7\pi. \tag{A}$$

(in all the preceding and subsequent, lines, k_1, k_2, k_3,... denote integers). C. Haselgrove denotes in his tables [1] O(t) to the expression

$$O(t) = \arg\{\pi^{-it/2}\} + \arg\{\Gamma(\frac{1}{4}+i\frac{t}{2})\}$$

and points out that it can be numerically evaluated from its asymptotic expansion

$$O(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\beta}{2} + \frac{1}{2}\sum_{r=1}^{\infty}\frac{B_r(1-2^{1-2r})}{2r(r-1)}.\frac{1}{t^{2r-1}},$$

but in the range $0 < t < 100$ we can dispense from this calculus, as he gives, in table I, the values of $\frac{1}{\pi}O(t)$.

With Haselgrove's notation, equation (A) turns out to be

$$O(t) + \arg\{\zeta(\frac{1}{2}+it)\} = \pm k_7\pi \tag{B}$$

which. of course, is valid for all real values of t.

The numerical validity of equations (A) or (B) can be easily checked by means of his tables, as he gives in other table the values of the real and imaginary parts of $\zeta(\frac{1}{2}+it)$.

The values of the argument of $\zeta(\frac{1}{2}+it)\}$ can then be calculated by means of the formula

$$\arg\{\zeta(\frac{1}{2}+it)\} = \operatorname{arctg}\frac{\operatorname{Im}\{\xi(1/2+it)\}}{\operatorname{Re}\{\xi(1/2+it)\}} \pm k_8\pi.$$

We have thus independently all the ingredients that enter in formula (A).

The delaits of the numerical checks, and a graphical representation of equation (A) can be seen in [2], where also is given an alternative derivation of the formula.

In order to see how formula (A) determines the position of the zeros on the critical line, it is sufficient to remember that $\arg\{\zeta(\frac{1}{2} + it)\}$ has a jump of π every time that t crosses a zero ("argument principle"), as can also easily seen in the graph for $\arg\{\zeta(\frac{1}{2} + it)\}$. (Of course, the jump has the value π if the zero is a simple one; has the value 2π if the zero is a double one, etc.).

This fact enables us to get from identy (A) the condition that determines the zeros on the critical line. This is evidently:

$$\frac{1}{\pi}\{\arg\{\pi^{-it/2}\} + \arg\{\Gamma(\frac{1}{4} + i\frac{t}{2})\} + \arg\{\zeta(\frac{1}{2} + it)\}\} = \text{integer number} \qquad \text{(C)}$$

and there cannot exist any other kind of zeros on the critical line.

Otherwise stated, (C) is a necessary and sufficient condition in order that a given value of t can be a zero of the zeta function on the critical line. This has been checked numerically in [2].

<u>Years 1887-1918</u>

Between these years, Stieltjes [3], von Mangoldt [4], and Backlund [5] determine by means of contour integration, the value of $N(T)$, the quantity of zeros of the zeta function inside the critical strip $0 \leq \sigma \leq 1$ in the interval $0 \leq t \leq T$. The methods used by them are very similar.

The work of von Mangoldt is particularly impressive, as it is written in a Weierstrassian way where every thing is exhaustively proved, even in its minimal details. Needless to say, the three calculations lead to a common result, that we proceed to analyse.

For the sake of simplicity, we choose Backlind's calculation, as exposed in Titchmarsh's handbook [6].

We have, by Cauchy's theorem of residues:

$$N(T) = \frac{1}{2\pi i}\int_{C} \frac{\xi'(s)}{\xi(s)}\, ds$$

where C is a rectangular contour including the critical strip; more specifically, the rectangle with vertices 2, $2+iT$, $-1+iT$, -1.

Due to reasons of symmetry, we can write this as

$$N(T) = \frac{1}{\pi i}\int_{C_1} \frac{\xi'(s)}{\xi(s)}\, ds$$

where C_1 is the straight part from 2 to $2 + iT$, and then from $2 + iT$ to $1/2 + iT$.

This can be also written as:

$$\pi N(T) = \Delta_{C_1} \arg\{\xi(s)\}.$$

The calculation then runs as follows:

$$\pi N(T) = \Delta_{C_1} \arg\{s(s-1)\} + \Delta_{C_1} \arg\{\pi^{-s/2}\} + \Delta_{C_1} \arg\{\Gamma(s/2)\} + \Delta_{C_1} \arg\{\zeta(s)\}$$

$$= \arg\{(\tfrac{1}{2} + iT)(-\tfrac{1}{2} + iT)\} - \arg\{2.1\} + \arg\{\pi^{-1/4-iT/2}\} - \arg\{\pi^{-2/2}\} +$$

$$\arg\{\Gamma(\tfrac{1}{4} + i\tfrac{T}{2})\} - \arg\{\Gamma(1)\} + \arg\{\zeta(\tfrac{1}{2} + iT)\} - \arg\{\zeta(2)\}$$

$$= \arg\{-\tfrac{1}{4} - T^2\} + \arg\{\pi^{-iT/2}\} + \arg\{\Gamma(\tfrac{1}{4} + i\tfrac{T}{2})\} + \arg\{\zeta(\tfrac{1}{2} + iT)\}$$

$$= \pi + \arg\{\pi^{-iT/2}\} + \arg\{\Gamma(\tfrac{1}{4} + i\tfrac{T}{2})\} + \arg\{\zeta(\tfrac{1}{2} + iT)\}.$$

As $N(T)$ can take only natural values, this is equivalent to say that there is a zero in the upper part of the critical strip every time that

$$\frac{1}{\pi}\{\arg \pi^{-iT/2} + \arg \Gamma(\tfrac{1}{4} + i\tfrac{T}{2}) + \arg \zeta(\tfrac{1}{2} + iT)\} = \text{natural number.} \qquad \text{(D)}$$

But it is a known fact that if $\beta_\gamma + iT_\gamma$ is a zero of the zeta function, then $\beta_\gamma - iT_\gamma$ is also a zero.

Hence, we can generalize (D) to the whole critical strip by stating that there are zeros there every time that

$$\frac{1}{\pi}\{\arg \pi^{-iT/2} + \arg \Gamma(\tfrac{1}{4} + i\tfrac{T}{2}) + \arg \zeta(\tfrac{1}{2} + iT)\} = \text{integer number.} \qquad \text{(E)}$$

Of course, any formula for $N(T)$ must include the zeros on the critical line determined by equation (C).

But it is a fact that equation (C) coincides with equation (E): there are not extra terms in (E) that could account for other zeros.

Hence the Riemann hypothesis would turn out to be true.

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